Rock Mechanics Seminar Series 2010

4. Yielding and plasticity in soils

Concepts – notation

We're used to porosity – In soil mechanics specific volume & void volume is more common.

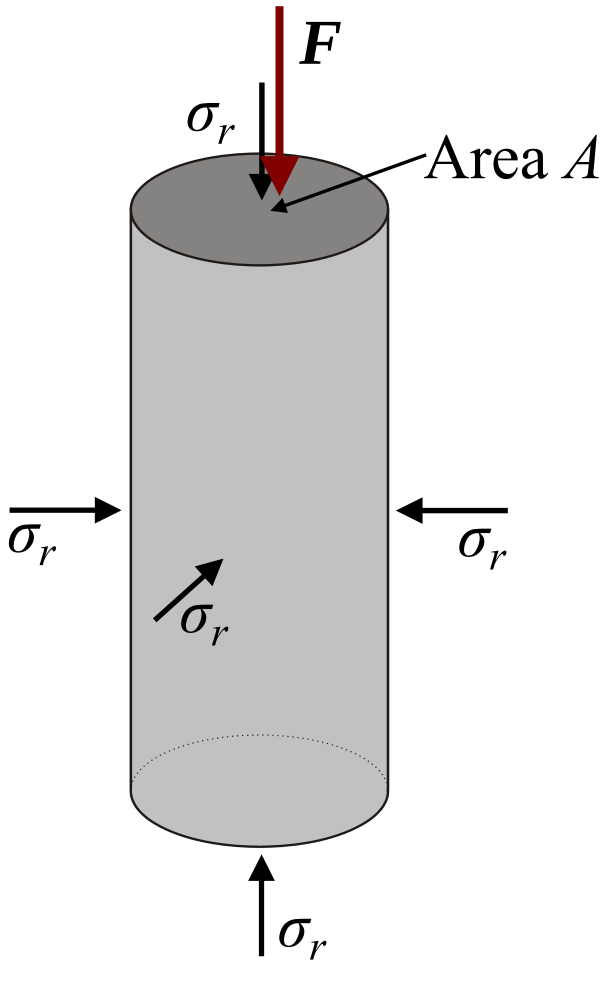
With bulk volume V_B, solids (grain) volume V_S, and pore volume V_P:
Specific volume $v = V_B/V_S$
Void volume $e = V_P/V_S$
As $V_B = V_S + V_P$,

$$\frac{V_B}{V_S} = 1 + \frac{V_P}{V_S} \Rightarrow v = 1 + e$$

$$\phi = \frac{V_P}{V_B} = \frac{V_B - V_S}{V_B} = 1 - \frac{1}{v}$$

$$\text{or} \quad v = \frac{1}{1-\phi}$$

Concepts – notation



Triaxial test

Sample (core) initially surrounded by fluid with pressure σ_r, uniformly acting on all sides (on nonpermeable membran)

Sample is then subjected to an axial force $\boldsymbol{F}$ acting on area A.

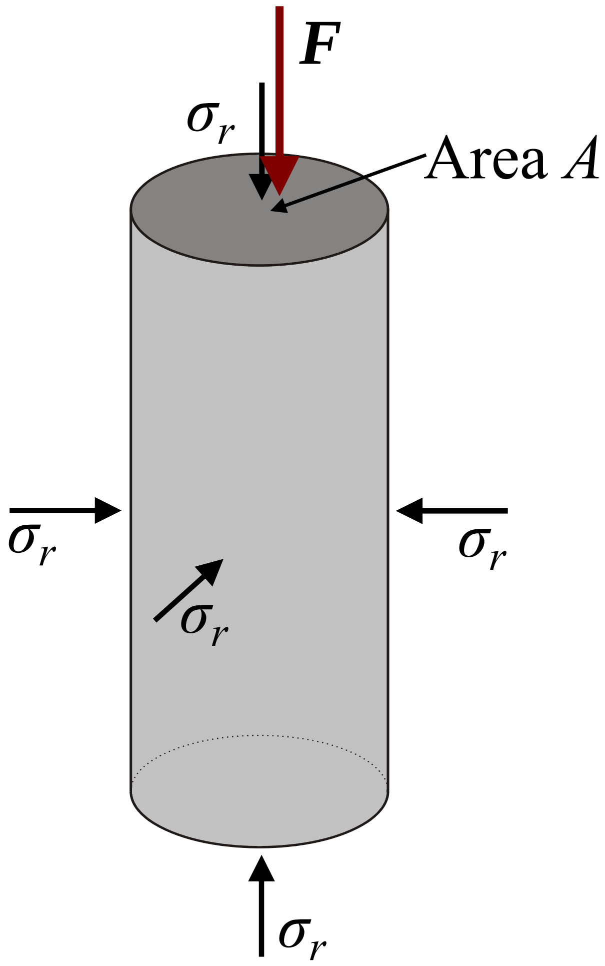
We measure $\boldsymbol{F}$, the elongation $\delta\ell$, and the volume change δV.

The axial stress is

$$\sigma_a = \frac{F}{A} + \sigma_r \text{, incremental: } \delta\sigma_a = \frac{\delta F}{A}$$

(Small correction term due to experimental setup neglected.)

Concepts – notation



Triaxial test

Deviator stress q:

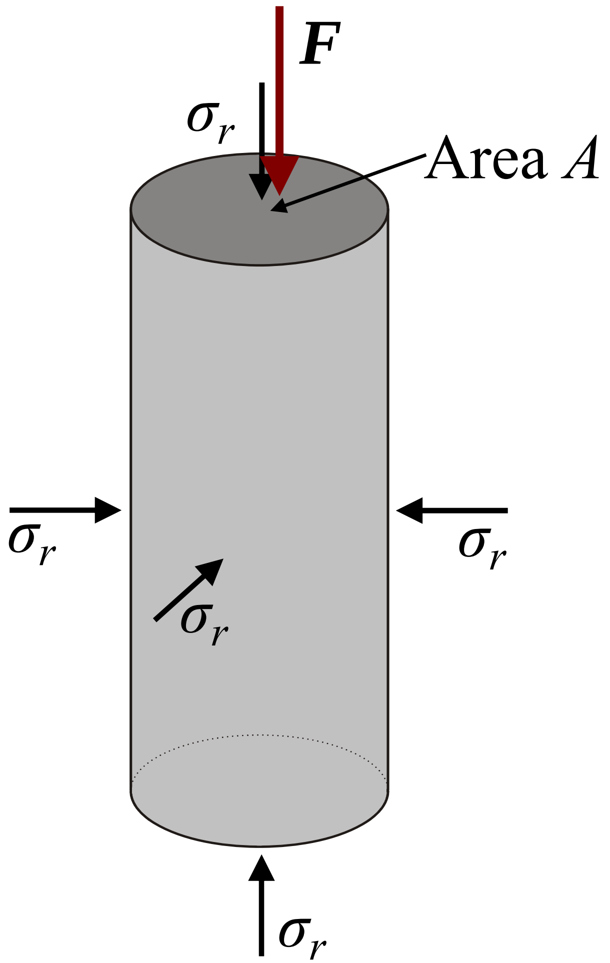
$$q = \sigma_a - \sigma_r \approx \frac{F}{A}$$

The sample may be saturated w. a fluid with pressure p_f. The relevant stress will henceforth be the effective stress **σ**', and I'll drop the " ' ". Hence unless explicitly stated **σ** will denote effective stress from now on.

The mean effective stress in this setup is:

$$p = \frac{\sigma_a + 2\sigma_r}{3}$$

Concepts – notation



Triaxial test

p and q will be our primary variables. (Partly / mainly because they are readily measureable in a triaxial test.)

Recall from seminar 1 that p and q in a non-axial-symmetry (general) setting were:

$$p = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

$$q = \{\frac{1}{2}[(\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + (\sigma_{xx} - \sigma_{yy})^2] + 3(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2)\}^{1/2}$$

Easier to view these as mean and deviator stress…

Yielding of clay

Recall our theoretical ”experiment” to determine a failure surface, and our inability to perform repeated experiments on a sample carried through to failure.

We can get pretty close by using clay samples, where ”identical samples” with ”identical” preloading history and ”identical” content & composition are (beleived to be) available.

Many such experiments have been reported, and the following theory is based on results from such studies.

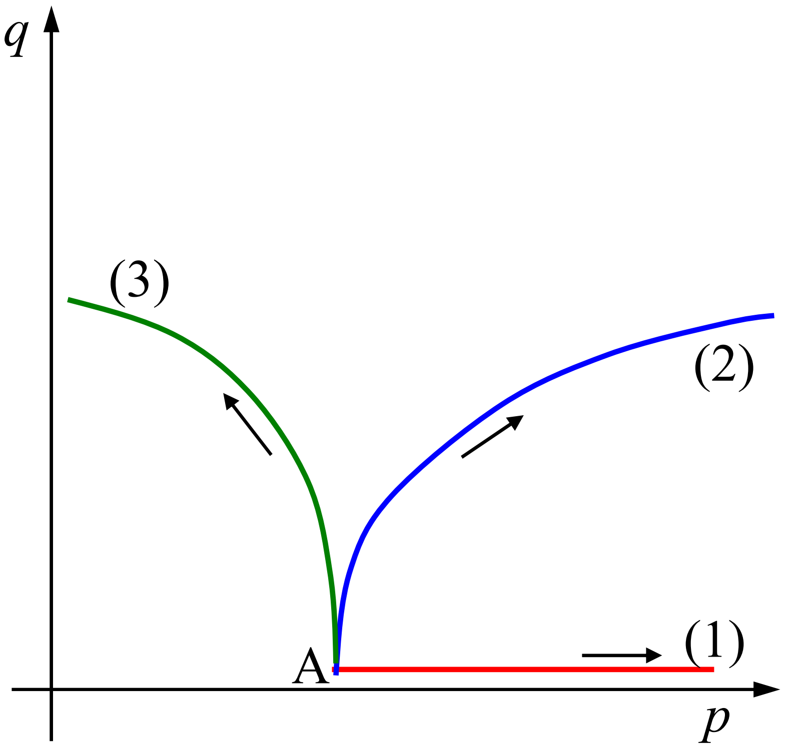
Yielding of clay

Triaxial clay test on three identical samples,
initial (p, q, v) are identical.
Experiment procedure (e.g.)

1. Isotropic compression (increasing cell pressure)
2. 1D compression – axial stress increased such that lateral strain does not occur
3. Conventional undrained compression test with pore pressure measurement.

The main idea is to perform loading tests with different stress paths.
In all experiments (p, q, v) are measured regularly.

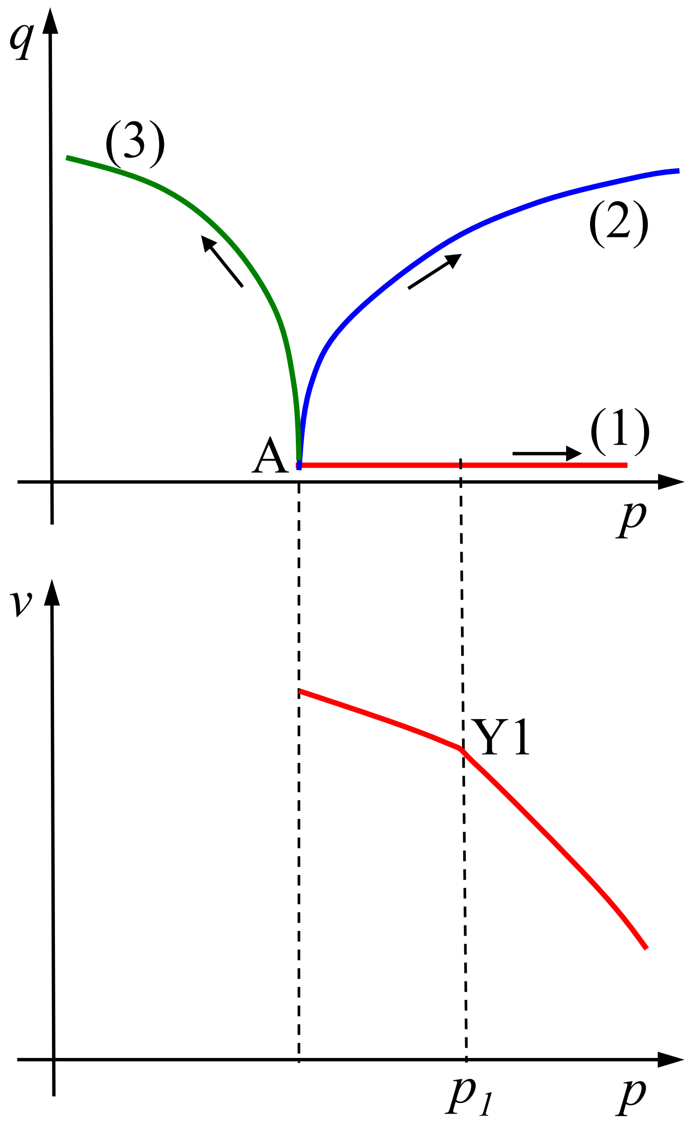
Yielding of clay



Results from the three different tests in (p:q)-space

All tests start in point A, but follow different stress paths (curves in (p:q)-space)

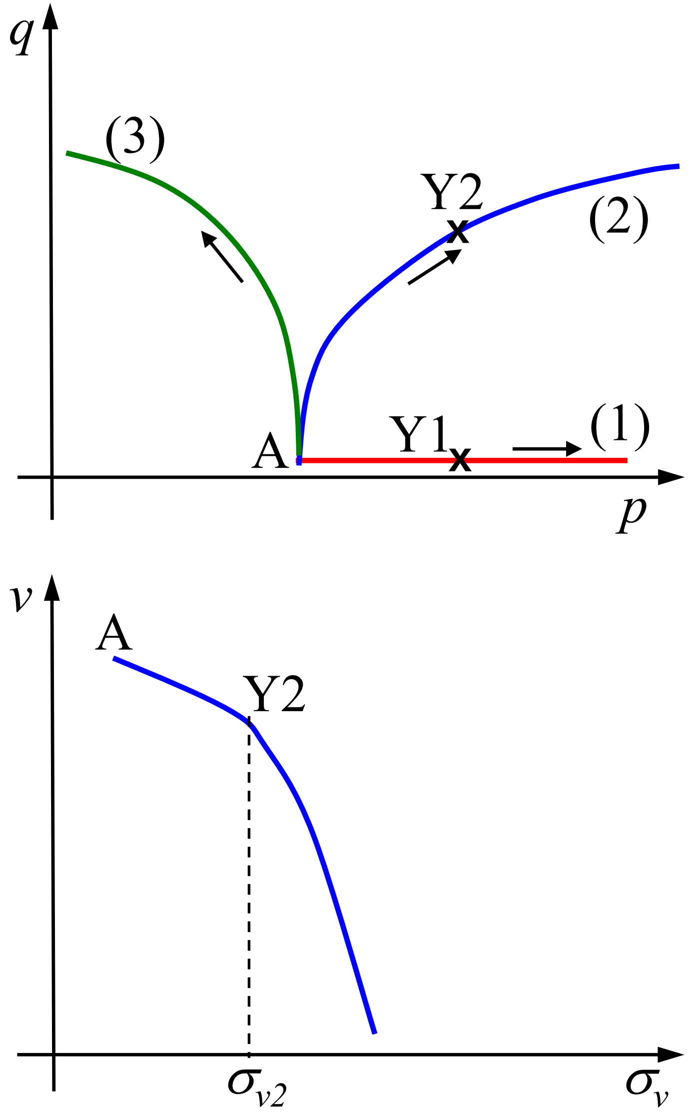
Yielding of clay



Experiment 1:
We observe the yield point Y1 at a *p*-value of p_1.

Can be marked on (*p*:*q*)-plot

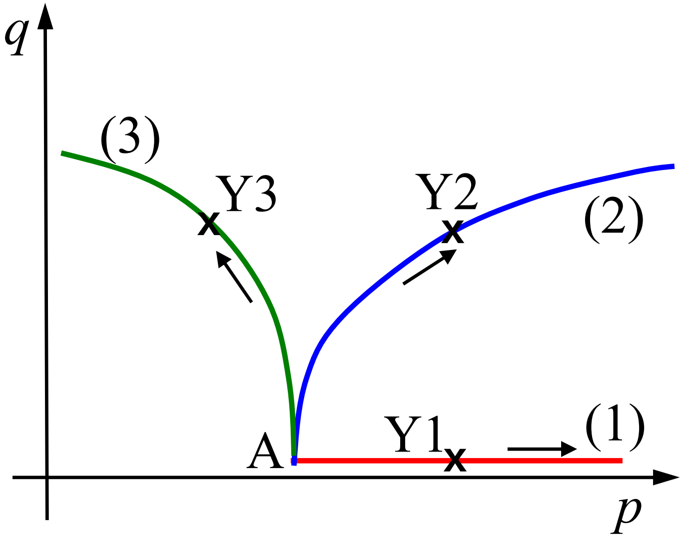
Yielding of clay

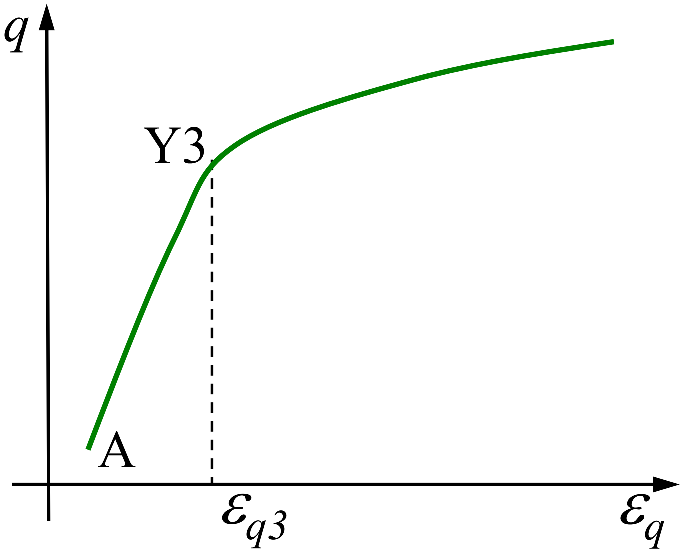


Experiment 2:
We observe the yield point Y2 at a vertical stress value of σ_{v2}.

Can be marked on (*p*:*q*)-plot

Yielding of clay

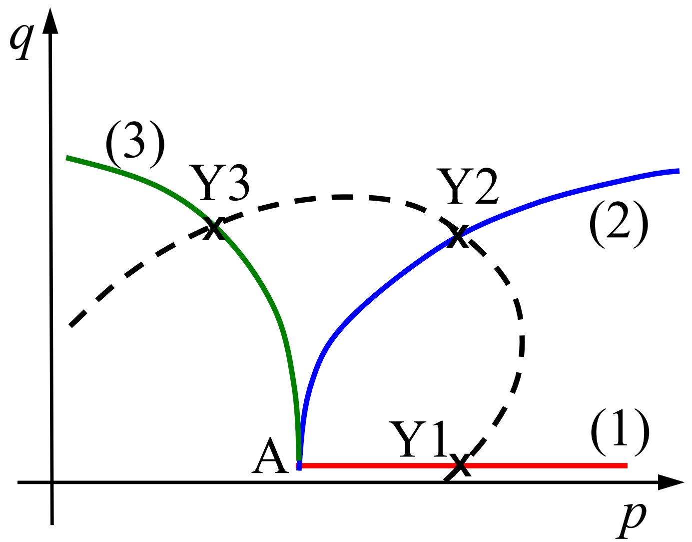




Experiment 3:
We observe the yield point Y3 at a shear strain value of ε_{q3}.

Can be marked on (p:q)-plot

Yielding of clay



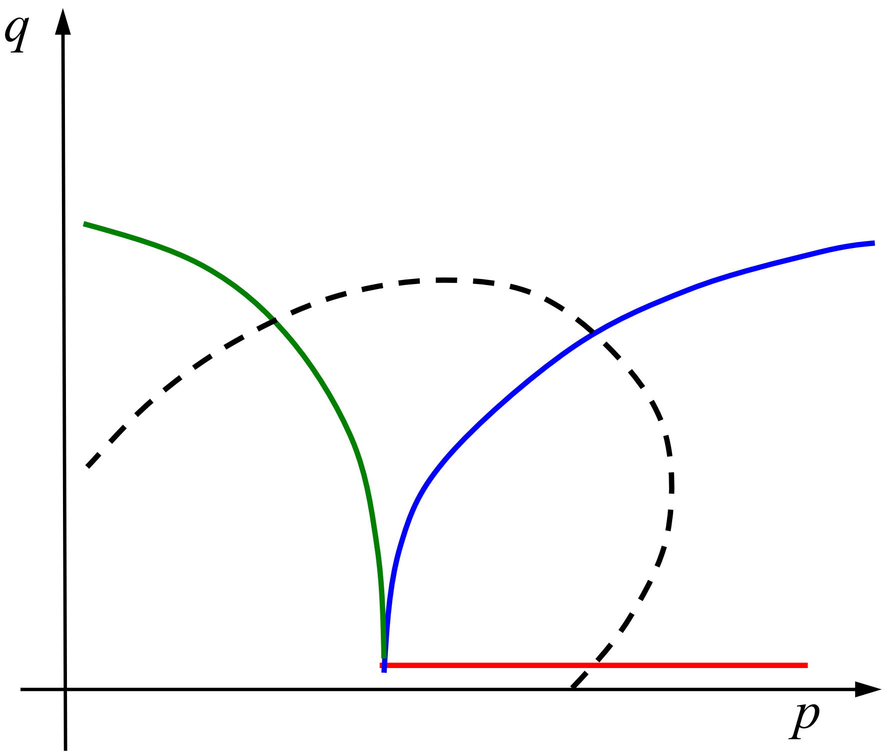
We already have a good idea of how to sketch the yield curve in (p:q)-space.

The main idea is to perform tests with different stress paths (curves in (p:q)-space), where also the p,q-values are readily measurable.

More advanced experiments could give yield surfaces in 3-D stress space, of which the (p:q)-plane is one particular section.

Additional tests could obviously have been done, increasing detail in the curve.

Yield surface



The yield surface (or yield locus) is a bound for all elastically attainable states for soil with one particular history.

Yield surface

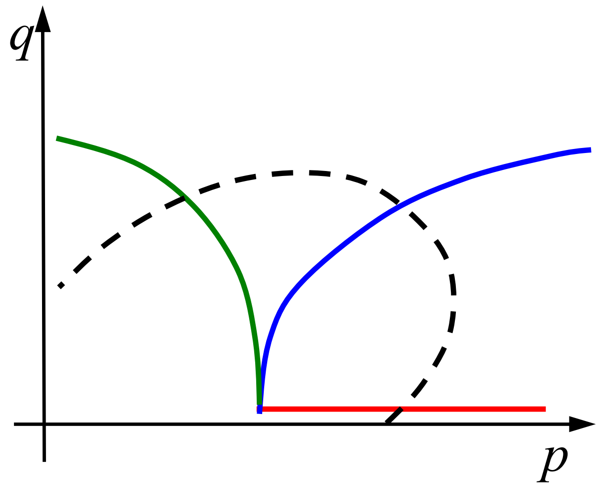
”Axiom”: A stress state can lie *on* or *inside* a current yield surface,
but never outside the surface.

”Passing” of a yield point requires the current yield surface to *change size, and possibly shape*.

Any subsequent probing investigates the shape of the new current yield surface, *not* the original yield surface.

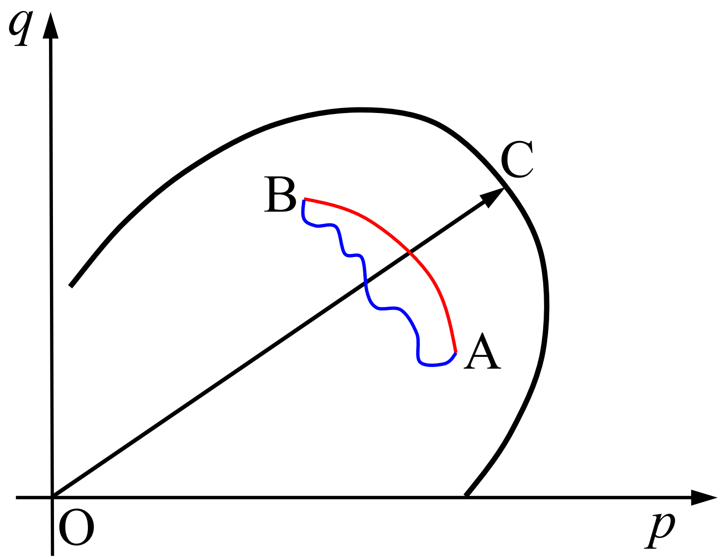
(& it was difficult to get *one* surface…)

Compare to ”intuitive” understanding of compaction in the grain pack model.



Elastic – plastic model for soil

Simplistic ”broad-brush” approach

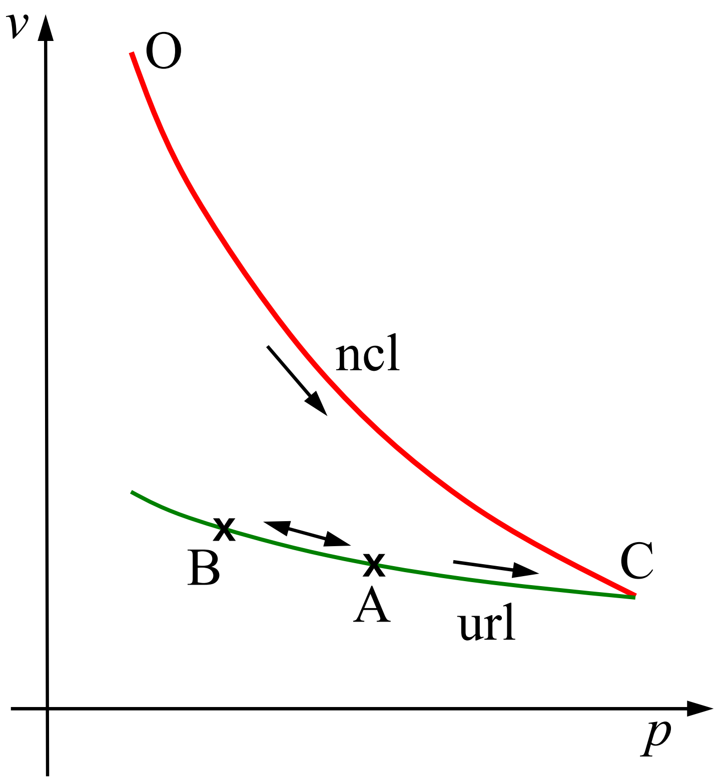


For this description we will assume that the changes in size of the current yield locus is related to volume changes. (”Volumetric hardening models”.)

In the elastic region changes are independent of the path (A to B and back along any path is indifferent). This is a characteristic of elasticity.

Establishing the current yield locus e.g. by OC is a result of the *past loading history*.

Elastic – plastic model for soil



The previous figure in (p:v)-space

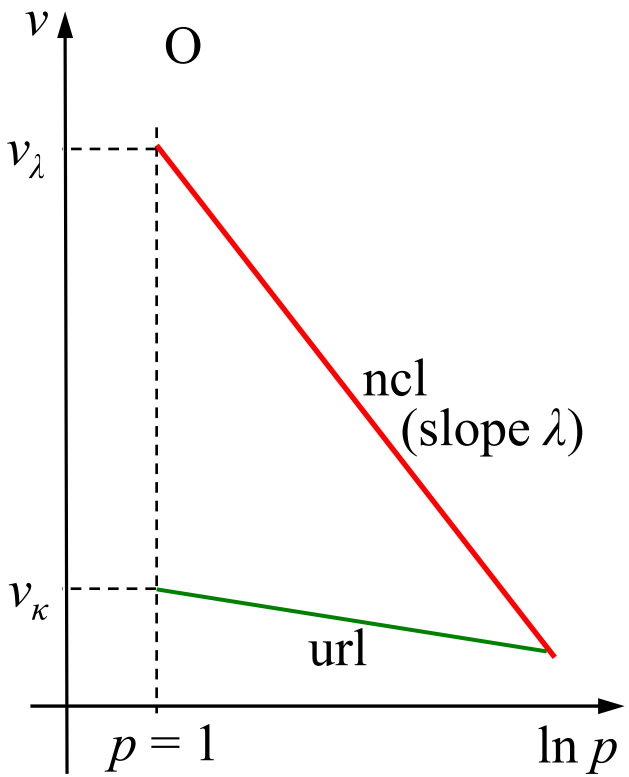
The expansion ("shape-shifting") of the current yield locus is along the *normal compression line* **(ncl).**

Changes within the elastic region are along *unloading – reloading lines* **(url)**.

Experimentally it has been found that the *ncl* often has a logarithmic form:

$$v = v_\lambda - \lambda \ln p$$

Elastic – plastic model for soil



ncl:

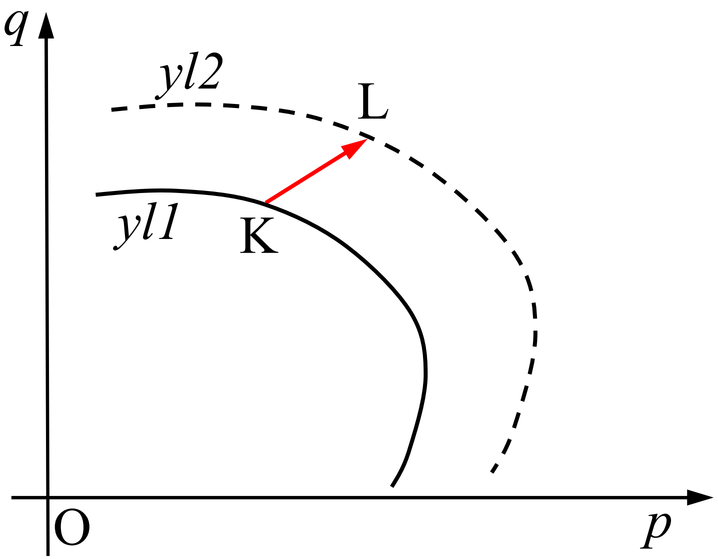
$$v = v_\lambda - \lambda \ln p$$

We also assume a similar form for the *url*:

$$v = v_\kappa - \kappa \ln p$$

(more questionable?)

Plastic volumetric strains & plastic hardening

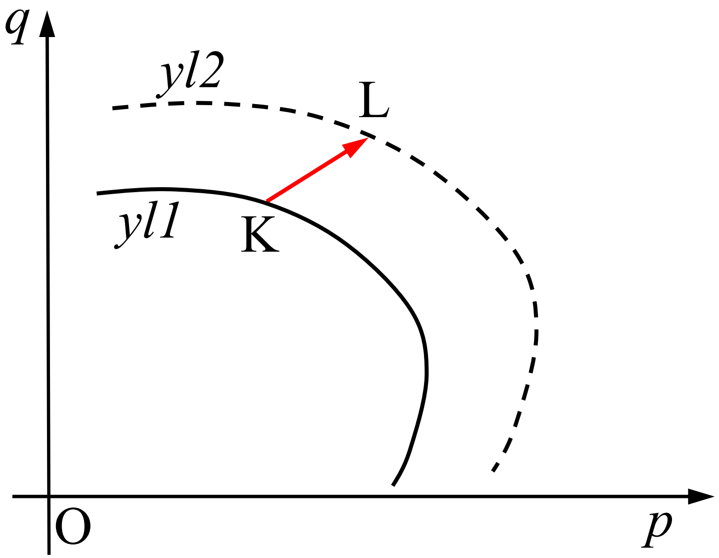


We are on a stress state **K** on the current yield locus *yl1*.
Moving to stress state **L** in (*p*:*q*)-space can only be achieved by the soil yielding (failing).
L must lie on a new yield locus *yl2*.

What is the shape of *yl2?*

Results from (the comparatively very few) experiments performed to determine the shapes of different yield loci on the "same" sample indicate that the shape does not change noticeably when the locus expands.
Based on scarce info & a desire to keep it simple, we state the axiom:
Irrespective of the stress path by which a new yield surface is created its shape remains the same.

Plastic volumetric strains & plastic hardening



Irrespective of the stress path by which a new yield surface is created its shape remains the same.

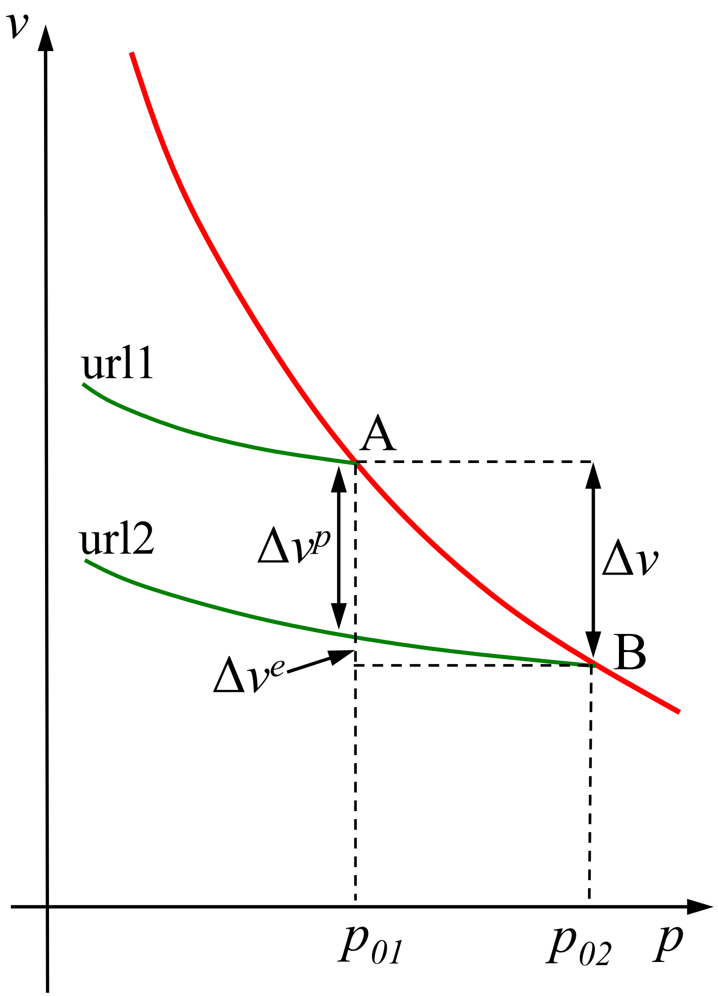
→ Convenient, but not necessary assumption

(but I don't think I've seen anyone *not* using it…)

The soil for which the yield locus is now *yl2* is opaque to attempts to elucidate details of its history.

(Note again how nicely this fits the intuitive understanding based on grain packing)

Plastic volumetric strains & plastic hardening



When reducing the volume from A to B and expanding the yield surface, the volume change is Δv,

$$\Delta v = \Delta v^e + \Delta v^p.$$

(elastic vol. chg + irrecoverable plastic vol. chg)

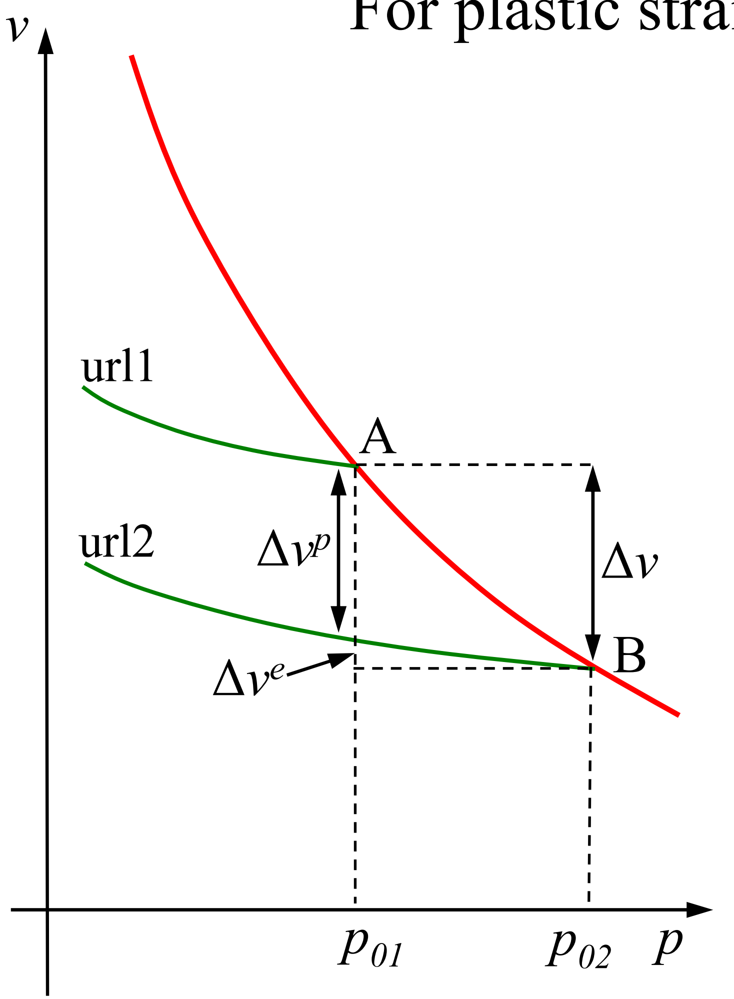
To get Δv^e we increase p from p_{01} to p_{02}, and then reduce it back to p_{01}. Using idealized *ncl* and *url*:

$$v^e = v_\kappa - \kappa \ln p \Rightarrow \delta v^e = \kappa \frac{\delta p}{p},$$

and from strain definition : $\delta \varepsilon_p = -\dfrac{\delta v}{v}$.

Hence $\delta \varepsilon_p^e = \kappa \dfrac{\delta p}{v^e p}$

Plastic volumetric strains & plastic hardening



For plastic strains:

$$\text{url1}: v = v_{\kappa 1} - \kappa \ln p$$

$$\text{url2}: v = v_{\kappa 2} - \kappa \ln p$$

$$\Rightarrow \Delta v^p = \Delta v_\kappa = v_{\kappa 2} - v_{\kappa 1}.$$

or

$$\begin{aligned}\Delta v^p &= v_\lambda - \lambda \ln p_{02} - (v_\lambda - \lambda \ln p_{01}) \\ &\quad + v_{\kappa 1} - \kappa \ln p_{01} - (v_{\kappa 2} - \kappa \ln p_{02}) \\ &= -\lambda \ln\left(\frac{p_{02}}{p_{01}}\right) + \kappa \ln\left(\frac{p_{02}}{p_{01}}\right) \\ &= -(\lambda - \kappa) \ln\left(\frac{p_{02}}{p_{01}}\right).\end{aligned}$$

For small changes, $\delta v^p = -(\lambda - \kappa)\dfrac{\delta p_0}{p_0}$

Plastic volumetric strains & plastic hardening

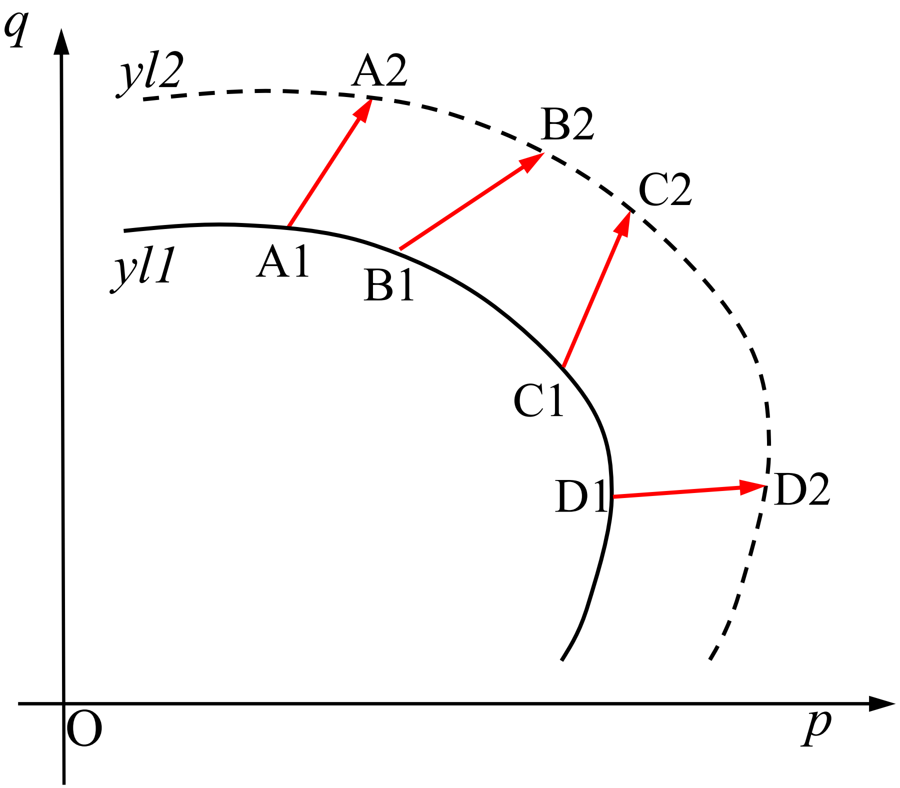
Then the plastic volumetric strain is :

$$\delta\varepsilon_p^p = (\lambda - \kappa)\frac{\delta p_0}{v p_0}$$

and the total volumetric strain increment

$$\delta\varepsilon_{\mathrm{p}} = \delta\varepsilon_p^e + \delta\varepsilon_p^p = \kappa\frac{\delta p}{v p} + (\lambda - \kappa)\frac{\delta p_0}{v p_0}$$

Plastic strain



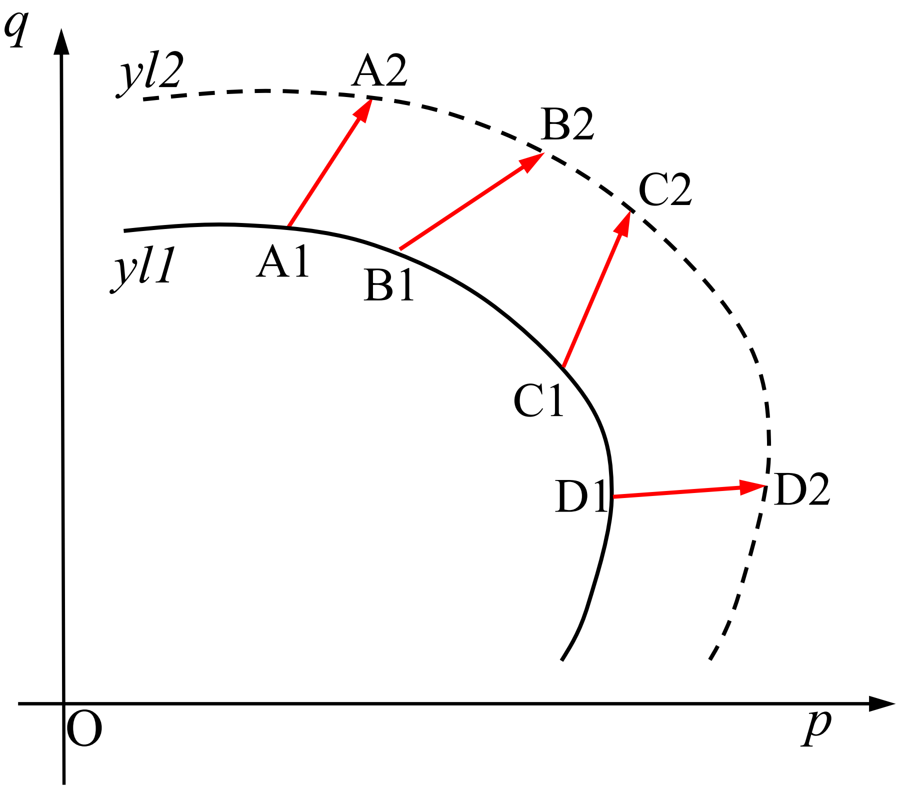
Expansion of yield locus from *yl1* to *yl2* could have been achieved by an infinitude of stress paths, e.g. as in figure: A1 – A2, B1 – B2, C1 – C2, D1 – D2.

Δp is different for these, hence also δv^e is different.

But as all stress paths link the same two yield loci, δp_0 and irrecoverable volume change is the same.

→ There is a basic difference in the way elastic and plastic volume changes are generated.

Plastic strain

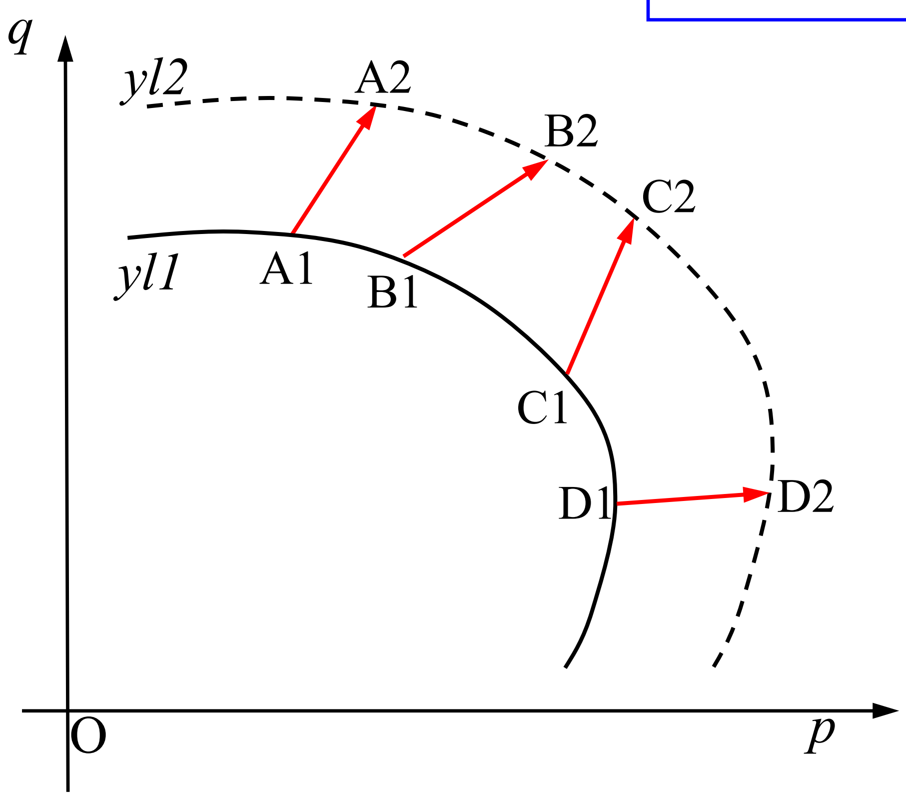


→ Plastic volumetric strains provide only a partial description of the plastic deformation – the magnitudes of the plastic *shear* strains also play a role.

The directions of the plastic strain increment vectors are governed by the particular combination of stresses at the particular point at which the yield surface was reached, *not* by the route through stress space that was followed to reach the yield surface.

Plastic strain

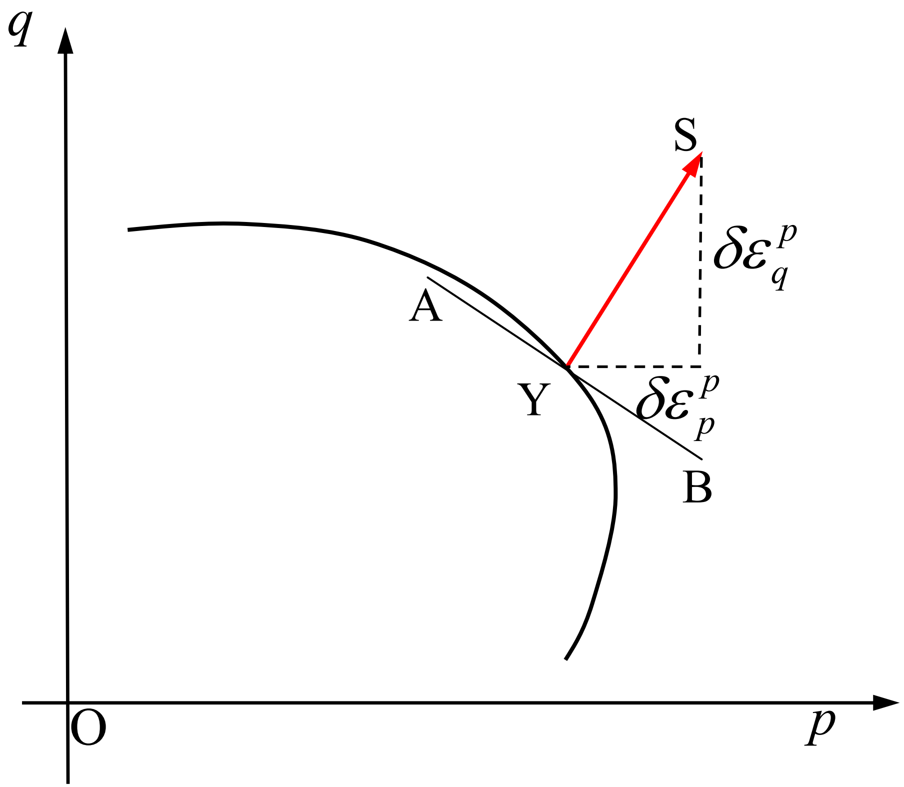
The relative magnitudes of strain increments are linked to:
Stresses – for plastic strain increments
Stress increments – for elastic strain increments



This is a distinguishing feature of plastic vs. elastic response.

Plastic potentials

Plastic deformations depend on the
stress state at which yielding occurs,
rather than on
the route by which that stress state is reached.



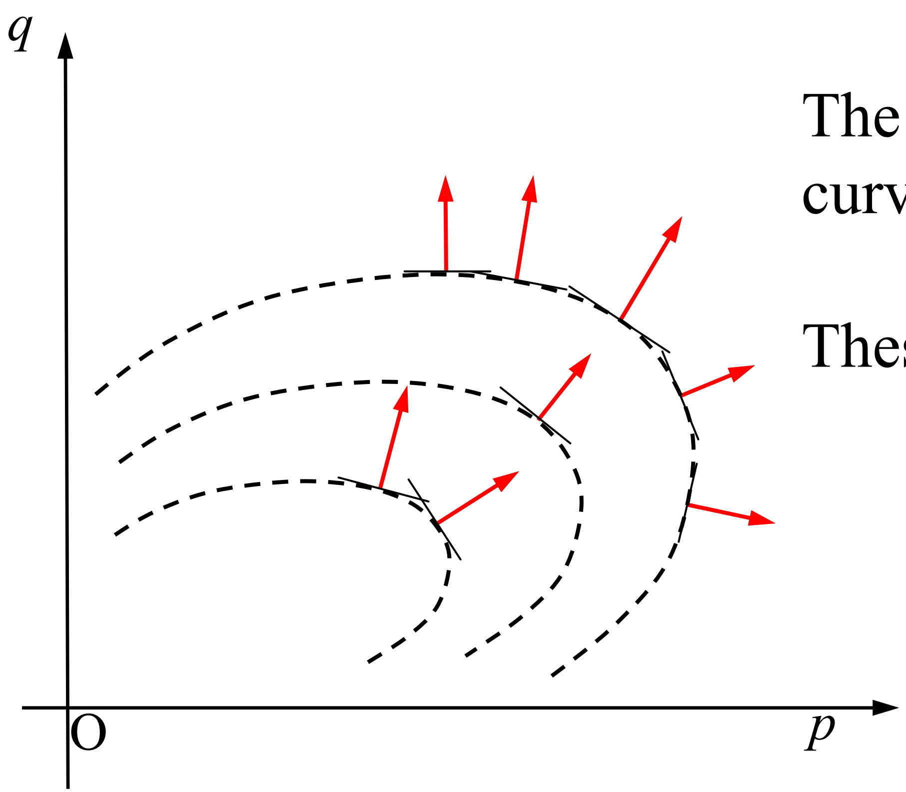
Yielding occurs at Y (in p:q-plane), associated with some irrecoverable volumetric plastic strain $\delta\varepsilon_p^p$ and some plastic shear strain $\delta\varepsilon_q^p$

Plot these components at Y to form a plastic strain increment vector YS.

Draw line AYB orthogonal to YS.

Plastic potentials

Yielding can occur under many combinations of stresses in the history of a soil.
For each combination, such a vector of plastic strain can be drawn through each yield point, as well as the normal (AB).

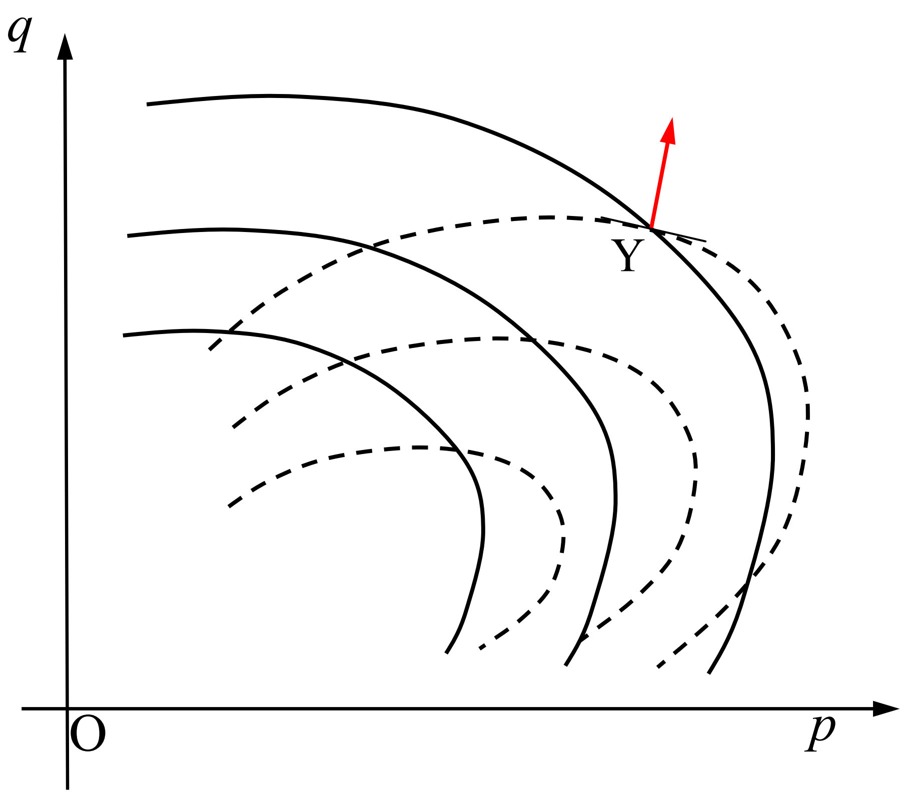


The set of such normals generate a family of curves which have the normals as tangents.

These curves are called **plastic potentials**

Given a plastic potential, it's normal at a point Y defines the *direction* of the plastic strain increment.
(*Magnitude* discussed earlier.)

Yield loci and plastic potentials



Yield loci (solid) &
plastic potentials (dashed)

Complete specification of the soil model requires information about both these sets of curves.

(Define magnitudes & directions)

Yield loci and plastic potentials

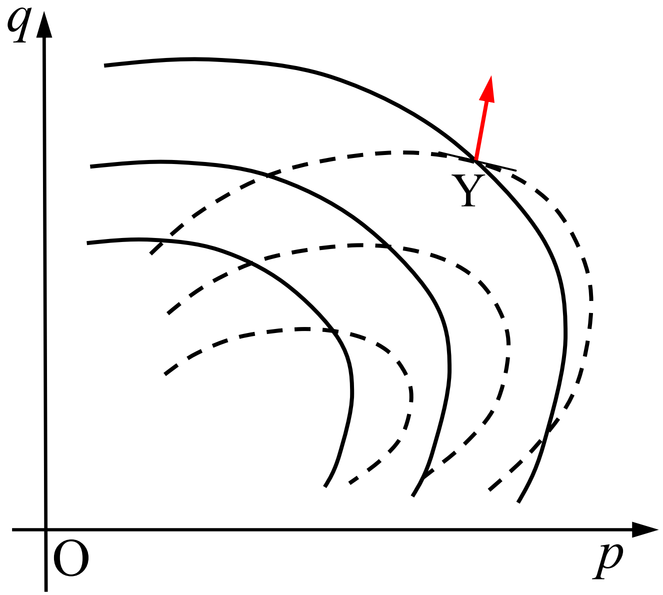
Obviously an advantage if yield loci and plastic potentials coincide.

If so, then the plastic strain increment vector is the outward normal to the yield surface.

Materials with such behaviour are said to obey the *postulate of normality*.

Another much used term for the same:
The material follows the law of
associated flow

(nature of plastic deformations is associated with the material's yield surface).



In general materials are described by *non-associated flow*.

Validity considerations

We've made quite a few assumptions.
These are convenient, but not necessary.

Pure theoretical considerations are more or less absent
– our models are, and must be based on experiments / observations.

Complex models with no / few assumption constraints do exist
(and are in use), but:
simplifying assumptions are most typically used due to
lack of empirical data.

General plastic stress – strain relationship

Having established an understanding of plasticity through a simplified material model, we are now in a good position to attack the general description:

Suppose the soil's yield loci are described by:

$f\,(p,\ q,\ p_0) = 0$ (p_0 indicates size of "current" locus)

and the plastic potentials:

$g\,(p,\ q,\ \zeta) = 0$ (ζ: parameter controlling size of potential passing through ($p,\ q$).)

General plastic stress – strain relationship

The plastic strain increments are related to the normal to the plastic potentials at current stress state:

$$\delta\varepsilon_p^p = \chi \frac{\partial g}{\partial p}; \qquad \delta\varepsilon_q^p = \chi \frac{\partial g}{\partial q}$$

χ is a scalar tied to the hardening characteristics (how far does the yield surface move in response to a stress change?)

Suppose change in yield locus size (Δp_0) is linked to increments in both plastic volumetric strain and plastic shear strain, i.e. a *hardening rule:*

$$\delta p_0 = \frac{\partial p_0}{\partial \varepsilon_p^p}\delta\varepsilon_p^p + \frac{\partial p_0}{\partial \varepsilon_q^p}\delta\varepsilon_q^p$$

General plastic stress – strain relationship

Differentiating yield loci expression:

$$\frac{\partial f}{\partial p}\delta p+\frac{\partial f}{\partial q}\delta q+\frac{\partial f}{\partial p_0}\delta p_0=0$$

Expressions on the last two slides can be combined and solved for χ:

$$\chi=-\frac{\dfrac{\partial f}{\partial p}\delta p+\dfrac{\partial f}{\partial q}\delta q}{\dfrac{\partial f}{\partial p_0}\left(\dfrac{\partial p_0}{\partial \varepsilon_p^p}\dfrac{\partial g}{\partial p}+\dfrac{\partial p_0}{\partial \varepsilon_q^p}\dfrac{\partial g}{\partial q}\right)}$$

which can be substituted back to obtained closed expressions for volumetric and shear plastic strain increments.