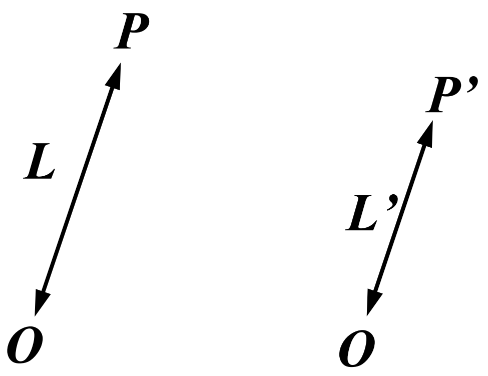
Rock Mechanics Seminar Series 2010

2. Strain, Elasticity

Definition: Strain ε – Elongation



Initial

After being subjected to stress.

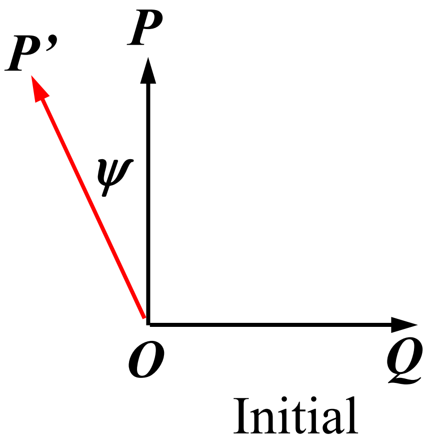
Line segment OP has been deformed to OP'.
The *elongation* ε is defined as;

$$\varepsilon = \frac{L - L'}{L}$$

Elongation is an example of strain:
Particles in a volume move in a manner which cannot be described by rigid motion or rotation of the volume as such.

Convention: $\varepsilon > 0$ for a contraction.

Definition: Strain ε – Shear



After being subjected to stress.

An initial orthogonal angle is deformed by an angle ψ. Then

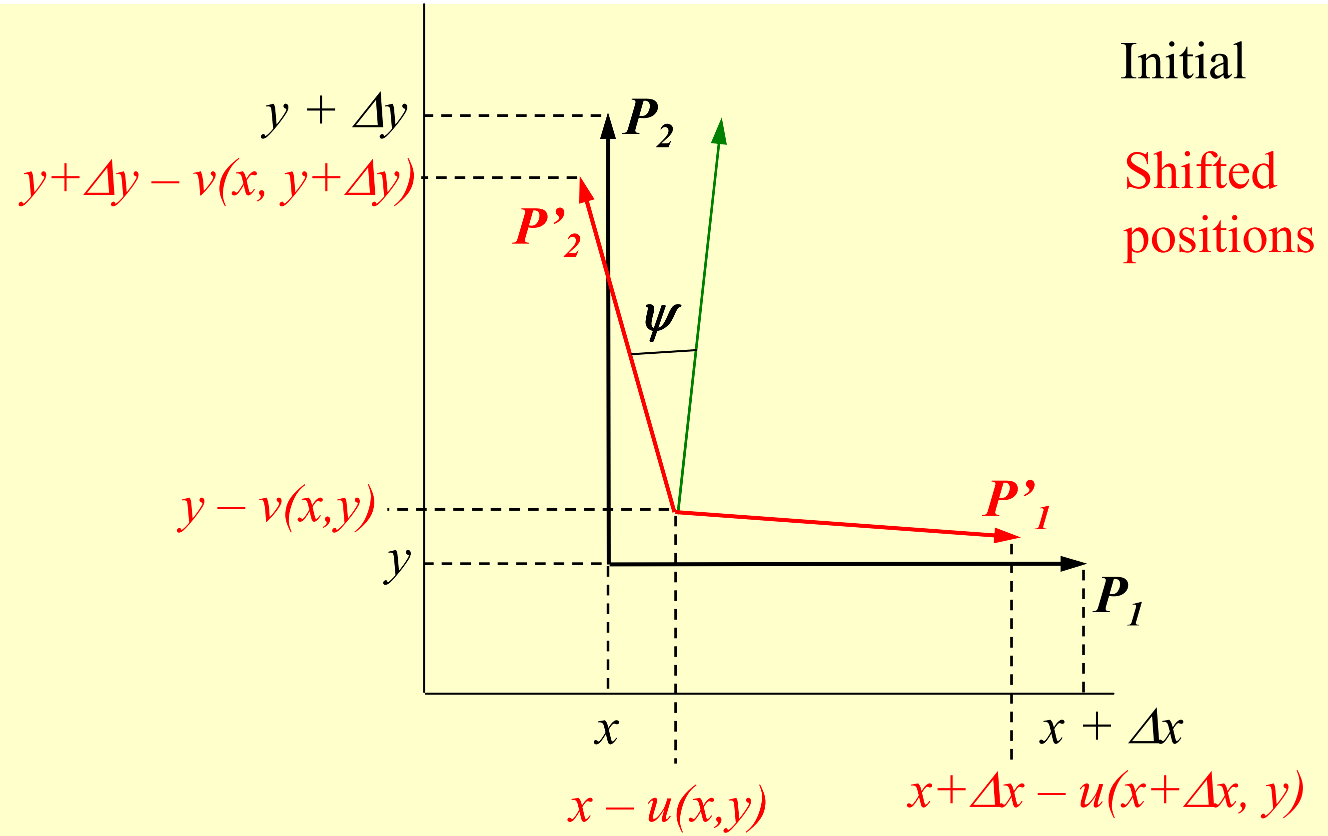
$$\Gamma = \frac{1}{2}\tan\Psi$$

is called shear strain corresponding to point O and direction OQ

In general, assume a particle (x,y,z) in a body is shifted to position (x', y', z') when the body is deformed. Set

$$x' = x - u$$
$$y' = y - v$$
$$z' = z - w$$

2D Infinitisemal strains



Elongation at *(x,y)*, in *x*-direction:

$$\varepsilon_x = \frac{(x+\Delta x) - x - [(x+\Delta x - u(x+\Delta x)) - (x - u(x))]}{(x+\Delta x) - x}$$

$$= \frac{u(x+\Delta x) - u(x)}{\Delta x} \to \frac{\partial u}{\partial x} \quad \text{when} \quad \Delta x \to 0$$

Shear strain corr. to *x*-direction:

$$\Gamma_{xy} = \frac{1}{2}\tan\Psi \approx \frac{1}{2}\sin\Psi$$

$$= -\frac{1}{2}\cos\left(\frac{\pi}{2} + \Psi\right)$$

$$= -\frac{1}{2}\frac{\mathbf{P'}_1 \cdot \mathbf{P'}_2}{|\mathbf{P'}_1| \cdot |\mathbf{P'}_2|}$$

$$\to \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

when $\Delta x, \Delta y \to 0$

All components of strain

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\Gamma_{xy} = \Gamma_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\Gamma_{xz} = \Gamma_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\Gamma_{yz} = \Gamma_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Strain Tensor:

$$\varepsilon = \begin{bmatrix} \varepsilon_x & \Gamma_{xy} & \Gamma_{xz} \\ \Gamma_{yx} & \varepsilon_y & \Gamma_{yz} \\ \Gamma_{zx} & \Gamma_{zy} & \varepsilon_z \end{bmatrix}$$

Volumetric Strain: (invariant)

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Alternative notation

$\mathbf{x} = (x, y, z)$
$\mathbf{u} = (u, v, w)$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1,2,3$$

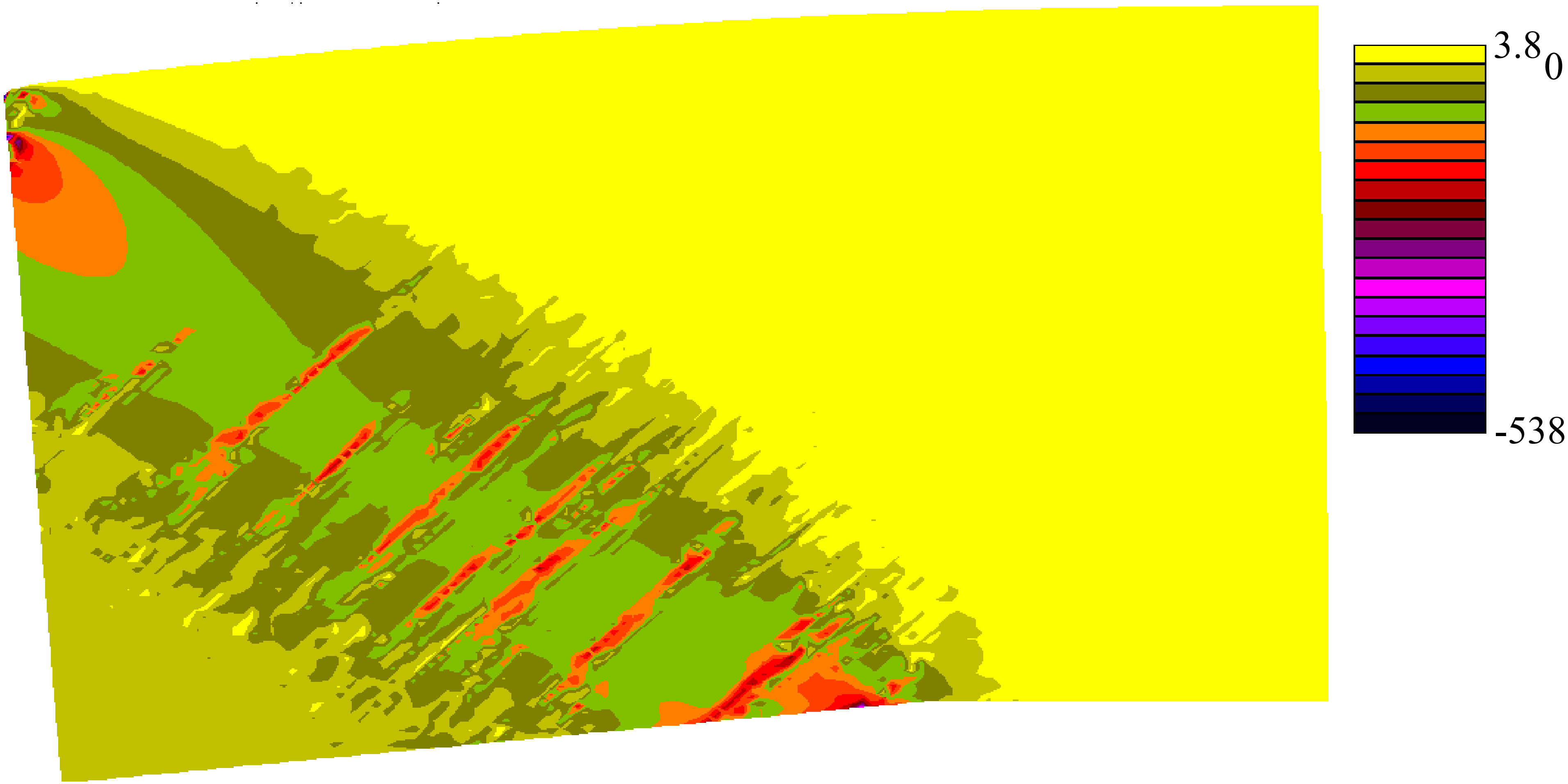
Obviously $\varepsilon_{ij} = \varepsilon_{ji}$

Strain Tensor:

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

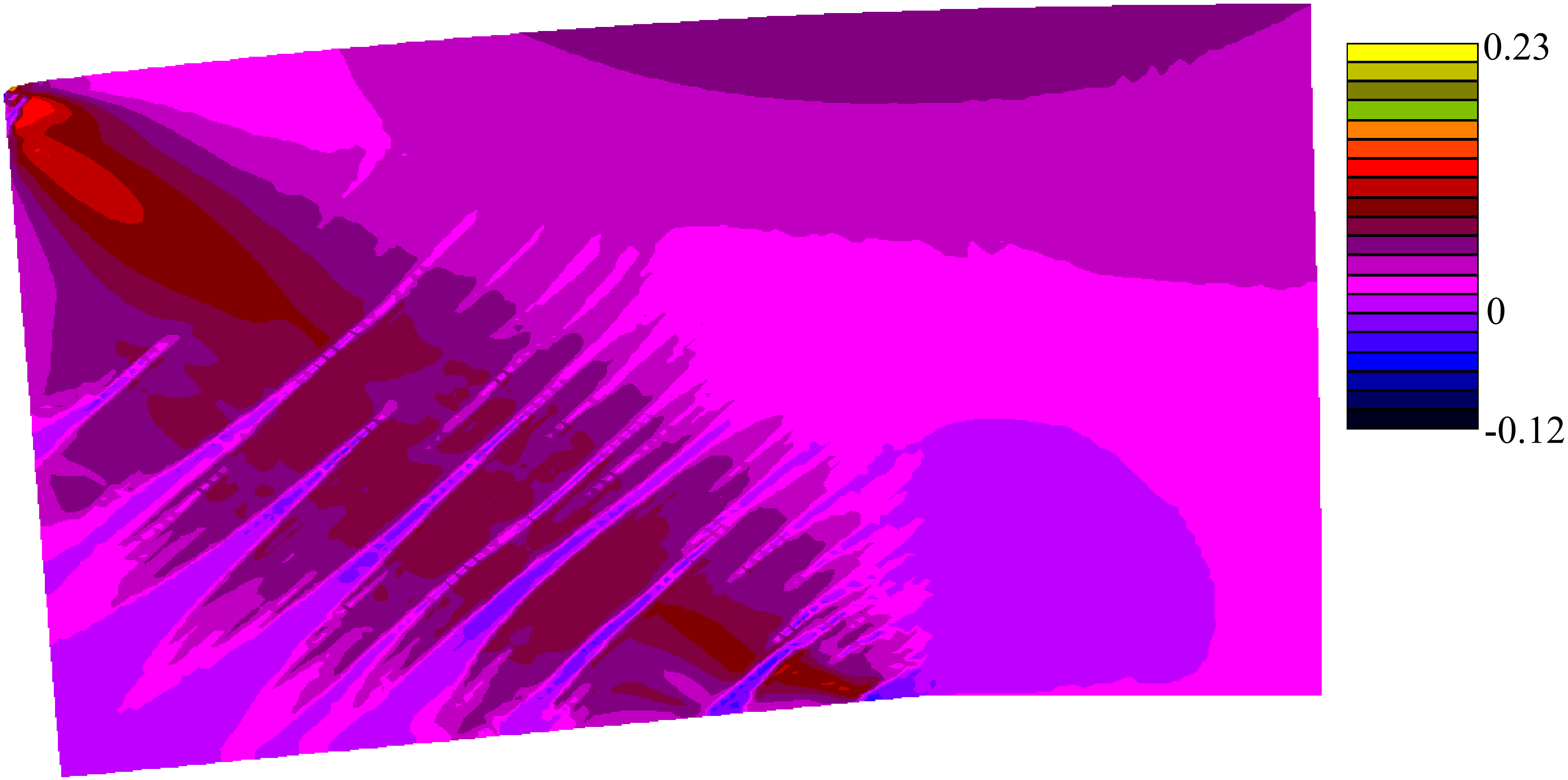
Symmetric, but no point in diagonalising if principal axes of strain $\neq$ principal axes of stress

Simulated bending of Sand Box (low tension strength)



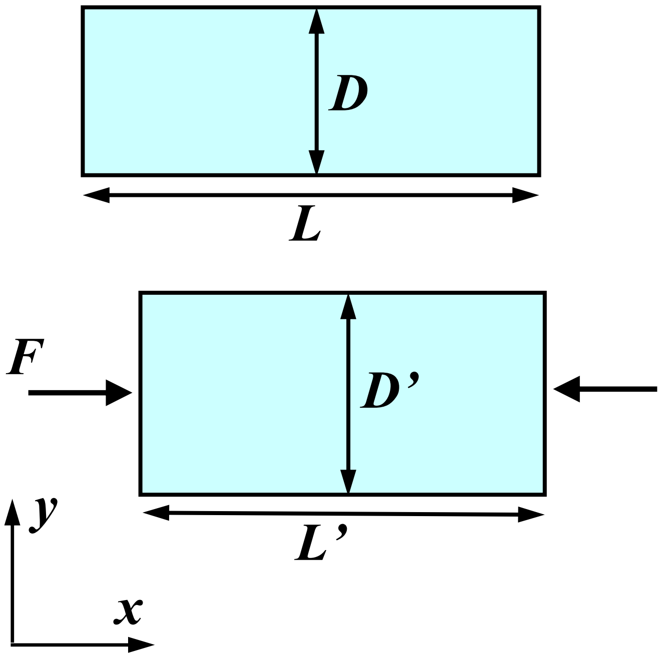
Mean effective stress (MPa)

Simulated bending of Sand Box (low tension strength)



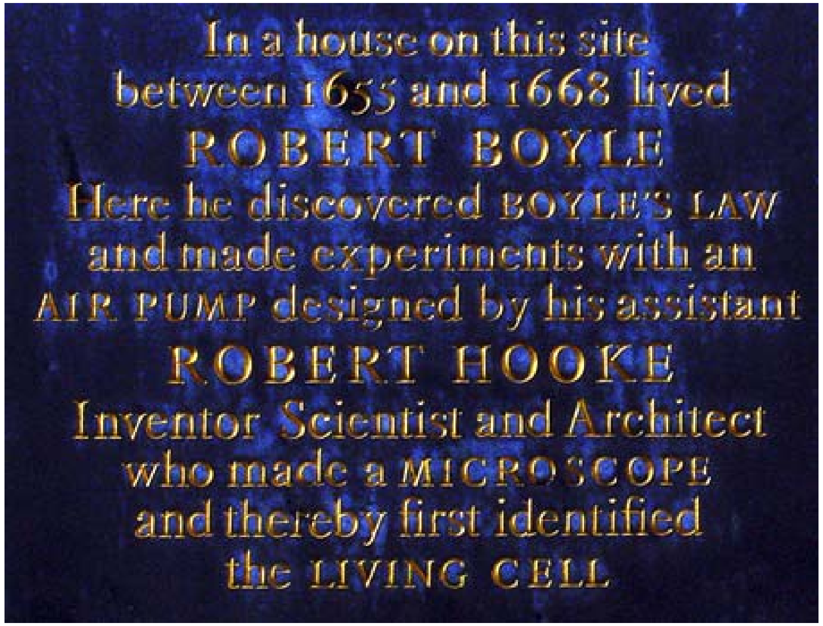
Volumetric Strain (MPa)

Linear Elasticity



Hooke's law:

$$\varepsilon_x = \frac{1}{E}\sigma_x$$



Memory from Oxford, ECMOR XII

Lateral elongation (width increase):

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

E (Young's modulus) and
ν (Poisson's ratio)
are examples of Elastic Moduli

Elastic Modili (in idealized experiments)

Young's modulus: $E = \frac{\sigma_{11}}{\varepsilon_{11}}$

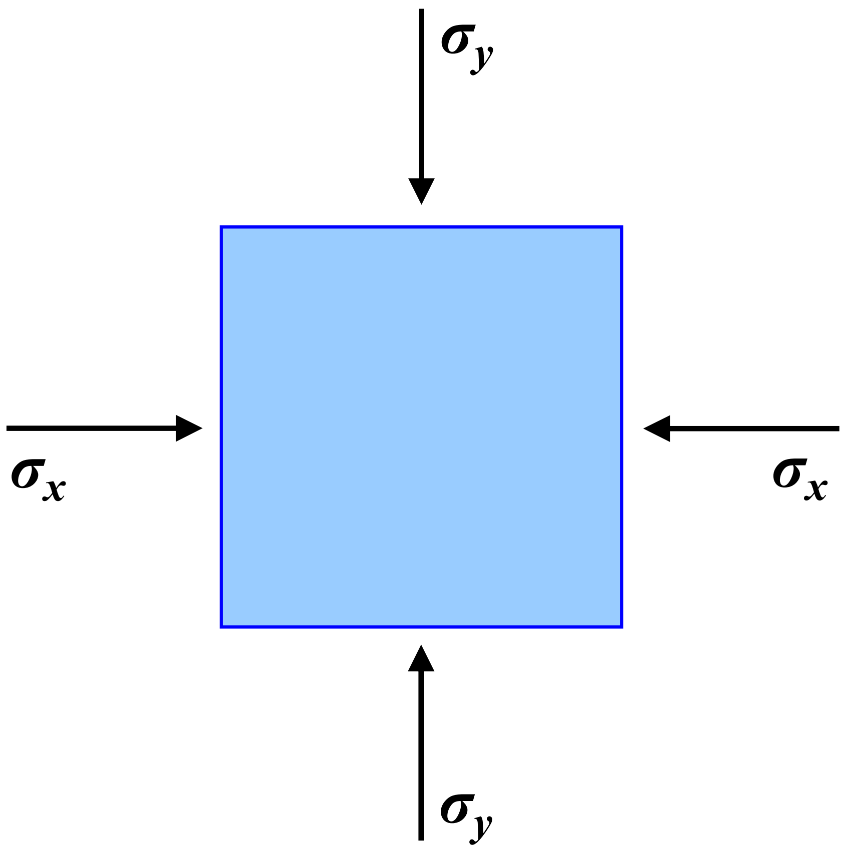
Poisson's ratio: $\nu = -\frac{\varepsilon_{33}}{\varepsilon_{11}}$

(for unidirectional stress, i.e. $\sigma_{11} \neq 0$, and $\sigma_{22} = \sigma_{33} = 0$)

An isotropic (and homogeneous) material will respond to applied stress independent of the orientation (of the stress).
For isotropic materials principal axes of stress and strain coincide.

Basic Constitutive Laws

Assuming infinitesimal displacement δ**u** the response is linear.



Total strain in x-direction =
= elongation strain caused by σ_x
+ width increase strain by σ_y:

$$\varepsilon_x = \frac{1}{E}\sigma_x + \varepsilon_x^{y-stress} = \frac{1}{E}\sigma_x - \nu\varepsilon_y$$

$$= \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y$$

Basic Constitutive Laws

Including z-direction:

$$\varepsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z$$

Normally expressed by using the mean stress $p \; (= \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z))$

$$\varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{3\nu}{E}p$$

$$\varepsilon_y = \frac{1+\nu}{E}\sigma_y - \frac{3\nu}{E}p$$

$$\varepsilon_z = \frac{1+\nu}{E}\sigma_z - \frac{3\nu}{E}p$$

$$\Gamma_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\Gamma_{xz} = \frac{1+\nu}{E}\sigma_{xz}$$

$$\Gamma_{yz} = \frac{1+\nu}{E}\sigma_{yz}$$

Basic Constitutive Laws

Solving these equations for stress:

$$\sigma_i = \frac{E}{1+\nu}\varepsilon_i + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_v, \ \ i = x, y, z$$

$$\tau_{ij} = \frac{E}{1+\nu}\Gamma_{ij}, \ \ ij = xy, xz, yz$$

Normally written as:

$$\sigma_i = 2G\varepsilon_i + \lambda\varepsilon_v, \ \ i = x, y, z$$

$$\tau_{ij} = 2G\Gamma_{ij}, \ \ ij = xy, xz, yz$$

where G and λ are other elastic modili.

Elastic Modili

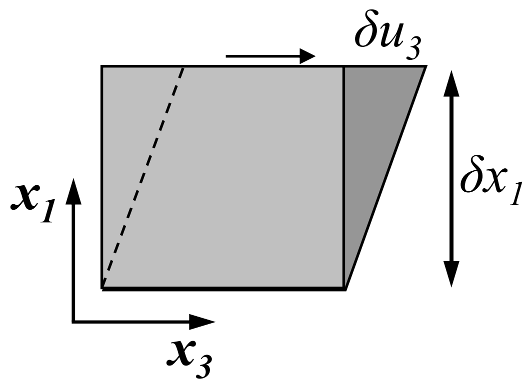
Young's modulus: $E = \frac{\sigma_{11}}{\varepsilon_{11}}$

Poisson's ratio: $\nu = -\frac{\varepsilon_{33}}{\varepsilon_{11}}$

Shear modulus[1)]: $G = \frac{1}{2}\frac{\sigma_{13}}{\varepsilon_{13}}$

Bulk modulus[2)]: $K = \frac{p}{\varepsilon_v}$

(Compressibility = *1/K*)



Shear strain: $\varepsilon_{31} = -\frac{\delta u_3}{\delta x_1}$

Lamé's constant λ: No direct physical meaning, but convenient. E.g.

$$K = \lambda + \frac{2G}{3} \text{ or } \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

[1)]Distortion w. no volume change. [2)]Volume change, no distortion

Elastic Modili – typical magnitudes

Material	E (GPa)	ν	K (GPa)	G (GPa)
Uncons. sand	0.01 – 0.1	~0.45		
Sandstone	0.1 – 30	0 – 0.45		
Clay	0.06 – 0.15	~0.4		
Shale	0.4 – 70	0 – 0.3		
Chalk, hiporo	0.5 – 5	0.05 – 0.35		
Chalk, loporo	5 – 30	0.05 – 0.3		
Granite	5 – 85	-0.3 – 0.4		
Calcite			74	27.5
Quartz			37.5	41

Elastic modili (cont'd)

In an isotropic (and homogeneous) material only two elastic modili are needed to fully describe the material.

A number of relationships between the five standard modili exist, and in practice the two modili that are most suited for the given experiment / situation will be used.

But unfortunately, isotropic materials are pretty rare in nature…

Anisotropic materials

In an anisotropic material nothing can be assumed.
E.g., Young's modulus will be direction dependent,
and the material's response to load (stress) will depend on
the stress orientation.

Still assuming linear behaviour, the most general stress–strain relationship will be:
Every component of stress depends on all components of strain:

$$\sigma_{ij} = \sum_{k,l} C_{ijkl}\varepsilon_{kl}$$

C_{ijkl} elastic constants. ($i, j, k, l = 1, 2, 3 \rightarrow 81$ constants)

Anisotropic materials (2)

$$\sigma_{ij} = \sum_{k,l} C_{ijkl} \varepsilon_{kl}$$

Symmetry considerations & no-rotation/translation at rest imply:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$$

Strain energy symmetry (not derived here):

$$C_{ijkl} = C_{klij}$$

$\rightarrow$ Number of independent constants reduced to 21

Anisotropic materials (3)

$$\sigma_{ij} = \sum_{k,l} C_{ijkl} \varepsilon_{kl}$$

Rocks normally possess orthorombic symmetry
(material has three perpendicular planes of symmetry)
Then the stress response will be identical if we e.g. let
x → x, y → y, z → -z
which implies

$$C_{1113} = C_{1123} = 0$$

(required to get identical stress response in transformed system)

Same argument on the other symmetry-transformations leave
9 surviving coefficients:
$C_{1111}, C_{2222}, C_{3333}, C_{1122}, C_{1133}, C_{2233}, C_{2323}, C_{1313}, C_{1212}$

Anisotropic materials (4)

$$\sigma_{ij} = \sum_{k,l} C_{ijkl} \varepsilon_{kl}$$

C_{1111}, C_{2222}, C_{3333}, C_{1122}, C_{1133}, C_{2233}, C_{2323}, C_{1313}, C_{1212}

Voigt notation: 4 indices $\rightarrow$ 2 indices:
11 $\rightarrow$ 1, 22 $\rightarrow$ 2, 33 $\rightarrow$ 3, 23 $\rightarrow$ 4, 13 $\rightarrow$ 5, 12 $\rightarrow$ 6

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

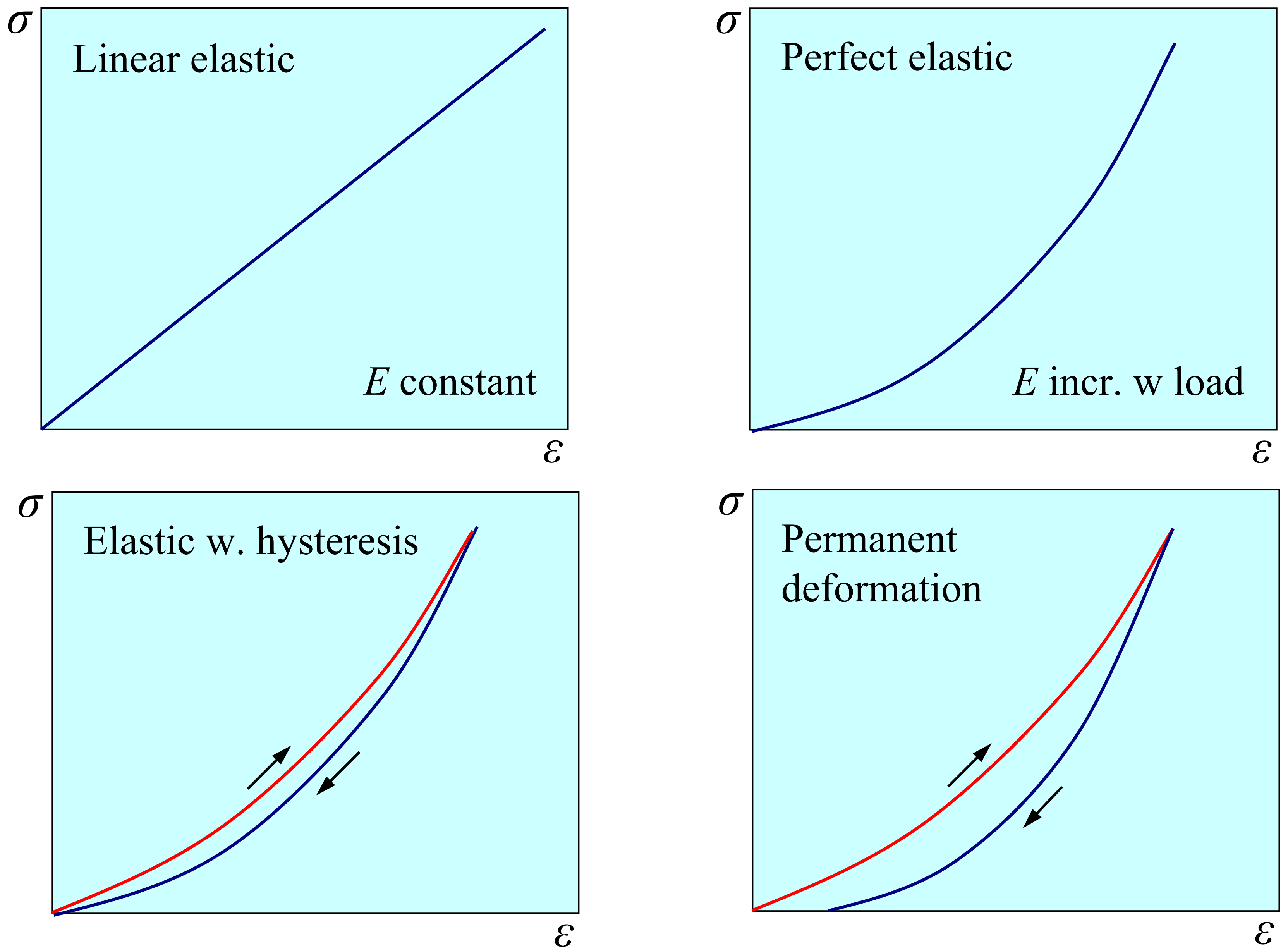
Anisotropic materials (5)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 2\Gamma_{yz} \\ 2\Gamma_{xz} \\ 2\Gamma_{xy} \end{bmatrix} \qquad \boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$$

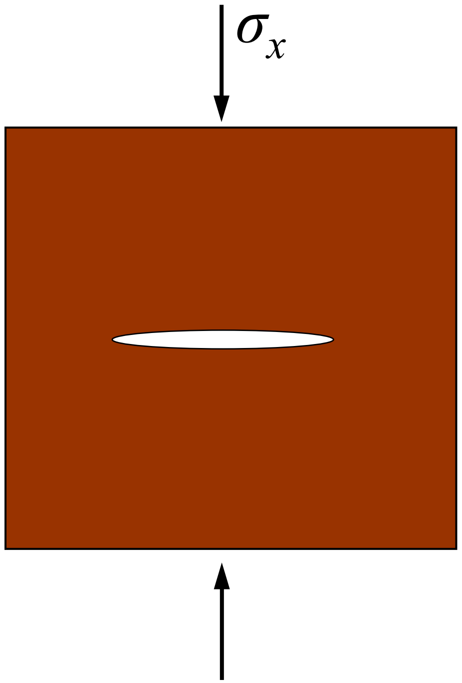
C is called the stiffness matrix, its entries are elastic constants. System of equations above generally describes most type of (linear elastic) rocks.

$\mathbf{S} = \mathbf{C}^{-1}$ is called the compliance matrix. $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}$

Nonlinear elasticity



Impact of cracks Ex. 1



Crack oriented w. face normal to stress.
No stress can be transferred across crack.
Hence the effective Young's modulus is reduced:

$$\frac{\sigma_x}{\varepsilon_x} = E_{eff} = E(1 - \xi Q)$$

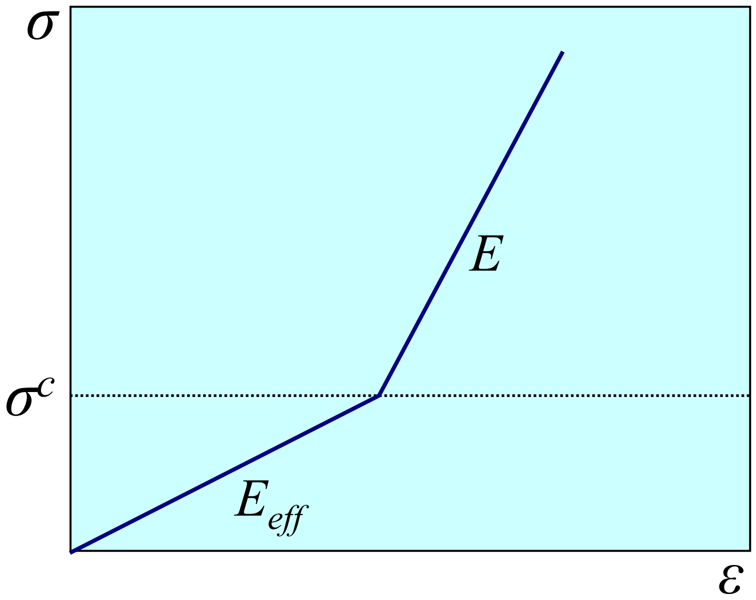
E: Young's modulus for equivalent uncracked material,
ξ: crack density
Q: crack shape factor

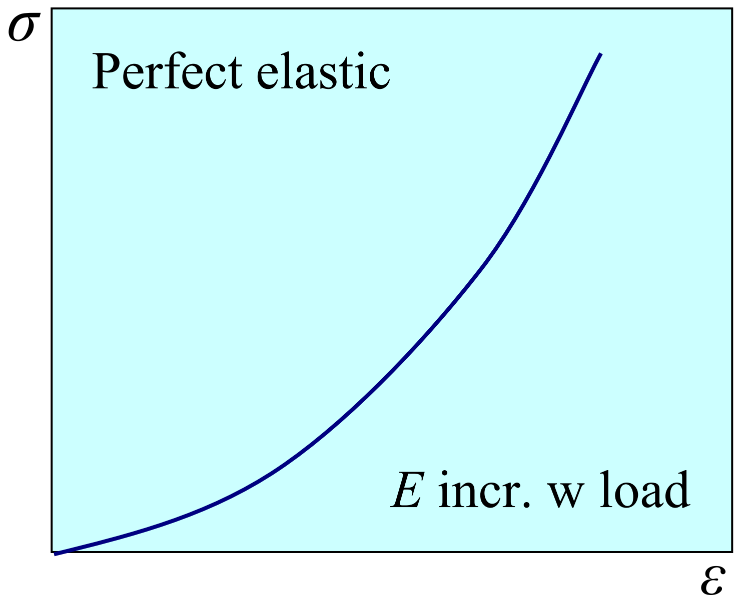
Increasing stress → ε_x increases. Part of strain increase caused by closure of crack. At stress level σ_x^c the crack closes, and for higher stress levels $\xi = 0$ and $E_{eff} = E$.

Impact of cracks Ex. 1

Qualitative stress – strain response:

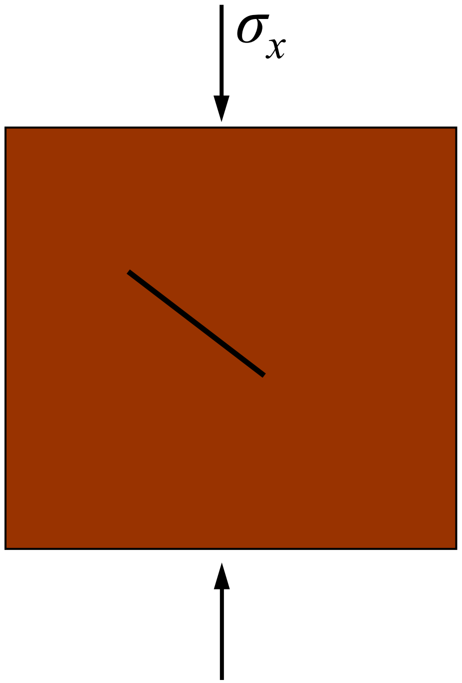
Compare to std. nonlinear curve





Right hand curve can actually be a process of continually closure of cracks.

Impact of cracks Ex. 2

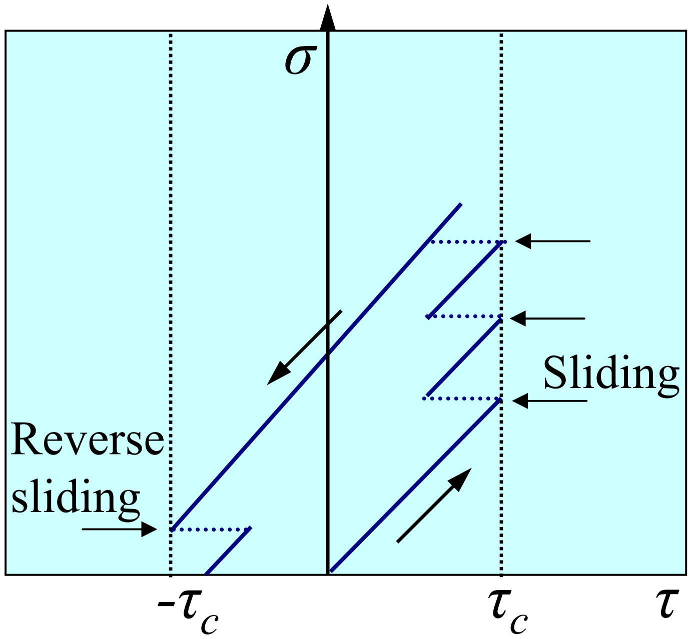


Closed crack oriented w. face at a finite angle to stress.
Due to friction the closed crack can transfer shear stress up to a certain level τ_c.
When τ exceeds τ_c the crack surfaces slip and slide, relieving stress, implying reduced τ below τ_c.
Sliding results in a strain increase.

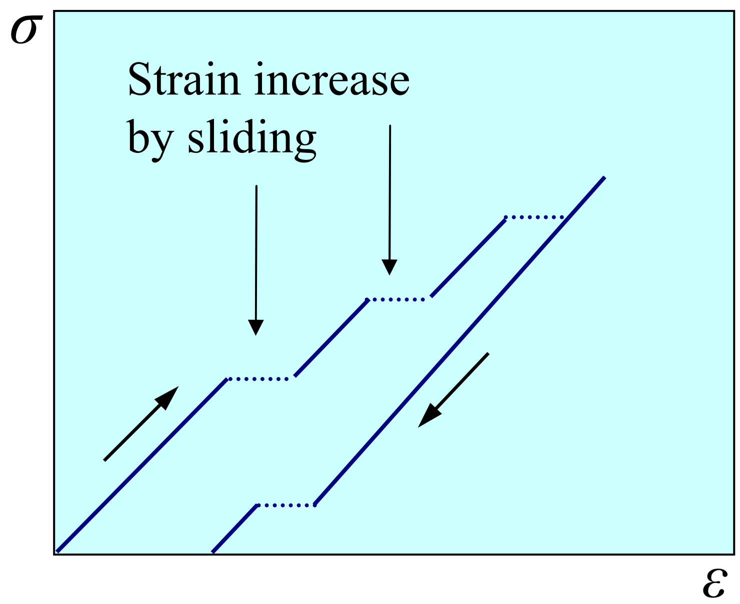
Upon unloading τ is reduced and may become equal to $-\tau_c$, inducing reverse sliding.

Impact of cracks Ex. 2

Normal stress – shear stress:



Stress – strain:

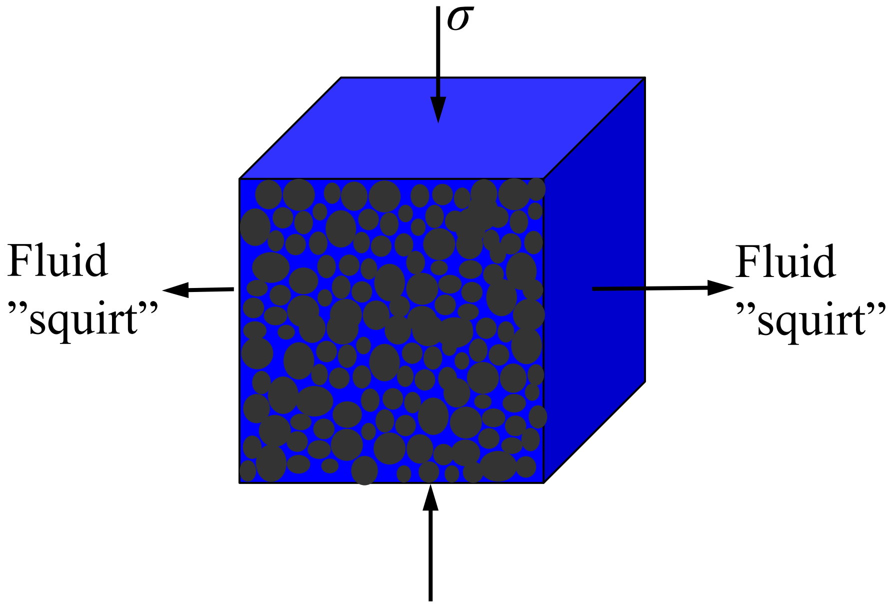


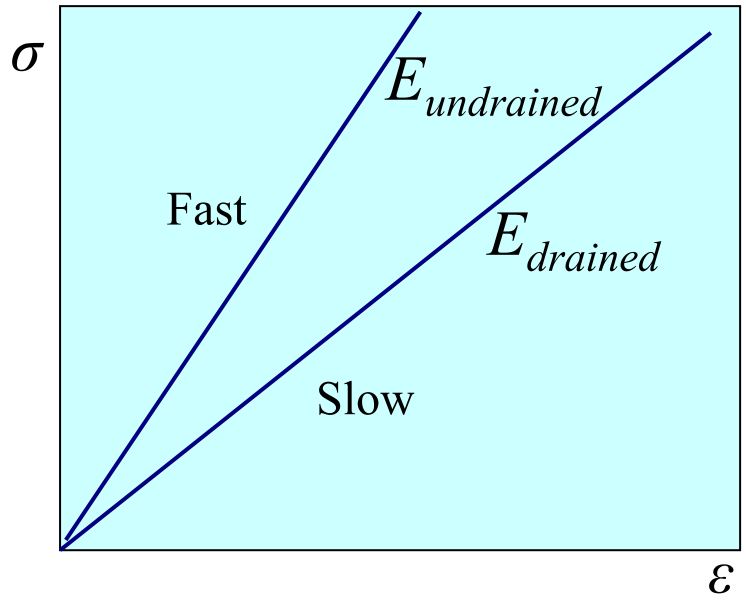
Compare to hysteresis-curve / permanent deformation.

→ Many nonlinear effects can be explained by behaviour of micro-cracks / impurities in material.

Poro-elasticity

Liquid saturated porous rock





If stress (load) increase is fast, the liquid won't have time to evade, and part of the applied stress will be carried by the fluid
→ apparently stiff rock.
For slow load increase the stress is taken up by the rock, and the fluid evades at static conditions → similar to dry rock, as discussed.

Basic Constitutive Laws: Poro-elasticity

$$\sigma_i = 2G\varepsilon_i + \lambda\varepsilon_v - C\varsigma, \;\; i = x, y, z$$

$$\tau_{ij} = 2G\Gamma_{ij}, \;\; ij = xy, xz, yz$$

$$p_f = C\varepsilon_v - M\varsigma$$

C and *M* are additional elastic modili required to describe the two-phase medium.
ζ is a strain parameter describing the volumetric deformation of the fluid relative to that of the solid:

$$\varsigma = \phi\nabla \cdot (\mathbf{u}_s - \mathbf{u}_f)$$

As the processes in oil / gas reservoirs are really *sloooowww*, we will not discuss the impact of fluid stress / strain any more.

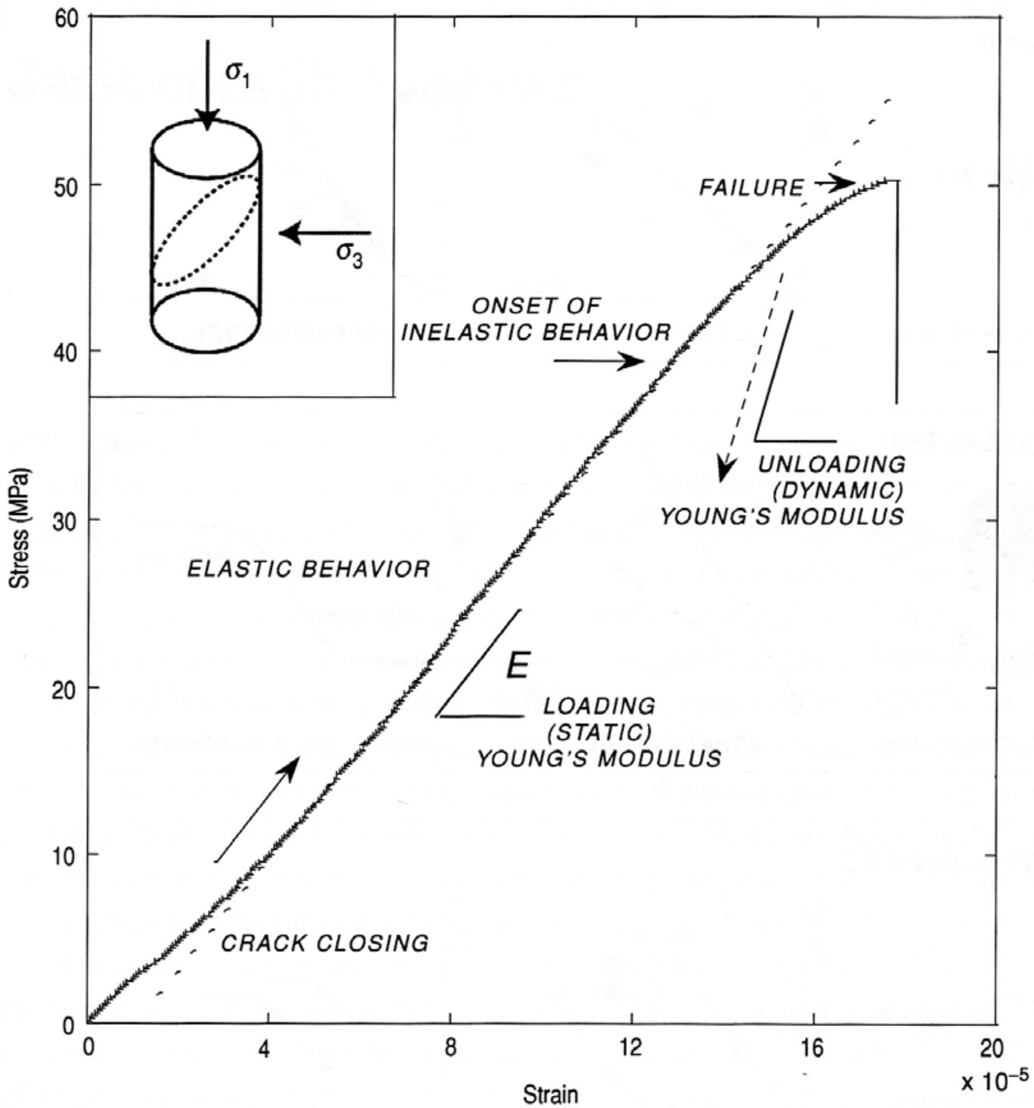
Real rocks / soils

Most if not all sands / weak sandstones (unconsolidated material) I've encountered don't behave elastic.

Zoback claims some well-cemented sandstone exhibits nearly ideal elastic behaviour over a wide range of applied stresses (next slide).

In general, most materials obey linear elastic laws only for small load increments.

Lab Experiment Well-cemented Sandstone



From Zoback

Increasing load:

1. Closure of micro-cracks (< 9 MPa)
2. Linear elastic (< 45 MPa)
3. Damaging of rock
   a) plastic deformation
   b) failure

Sets the scene for coming seminars…

Note that the strain values for this example are really small.
I.e. the rock is relatively strong, and strains are almost ”infinitesimal”