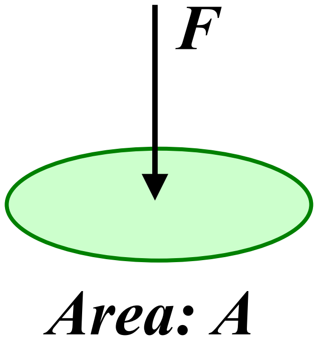
Rock Mechanics Seminar Series 2010

1. Stress

Definition: Stress σ

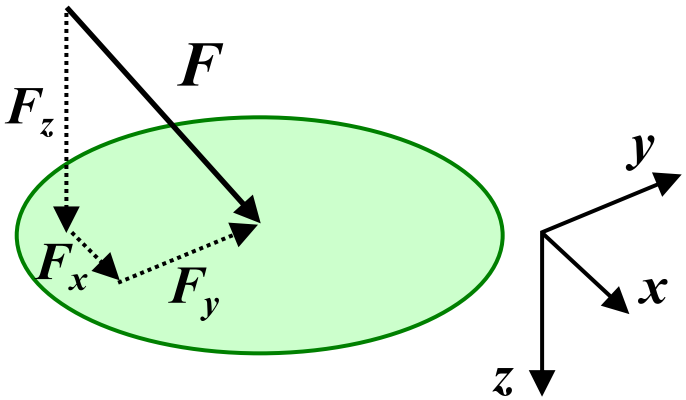


$\sigma = F/A$

(just like pressure.
However pressure is a scalar magnitude)

Units: Pa = N/m^2. (Or bars, atm, psi)
Often MPa or GPa.

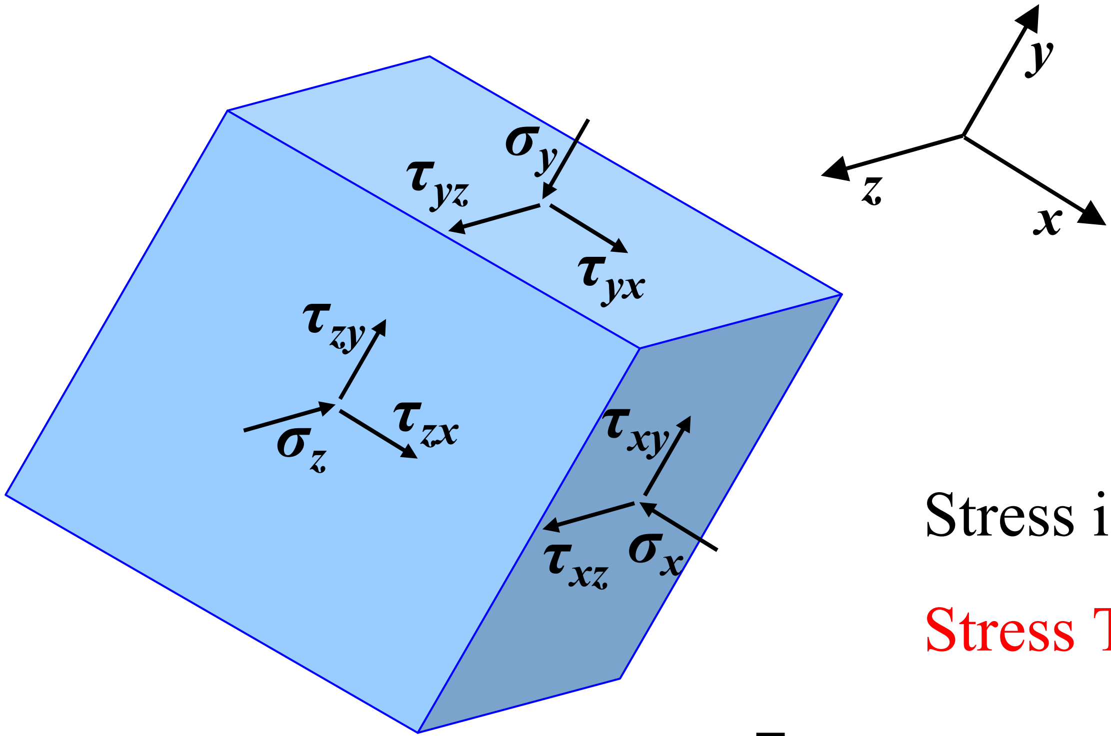
Stress isn't always normal



Perpendicular:
Normal stress: $\sigma = F_z/A$

Along surface:
Shear stress: $\tau_x = F_x/A$
$\tau_y = F_y/A$

Generalising to stress acting on elementary cube *dV*



Stress is completely described by

Stress Tensor:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Equilibrium considerations

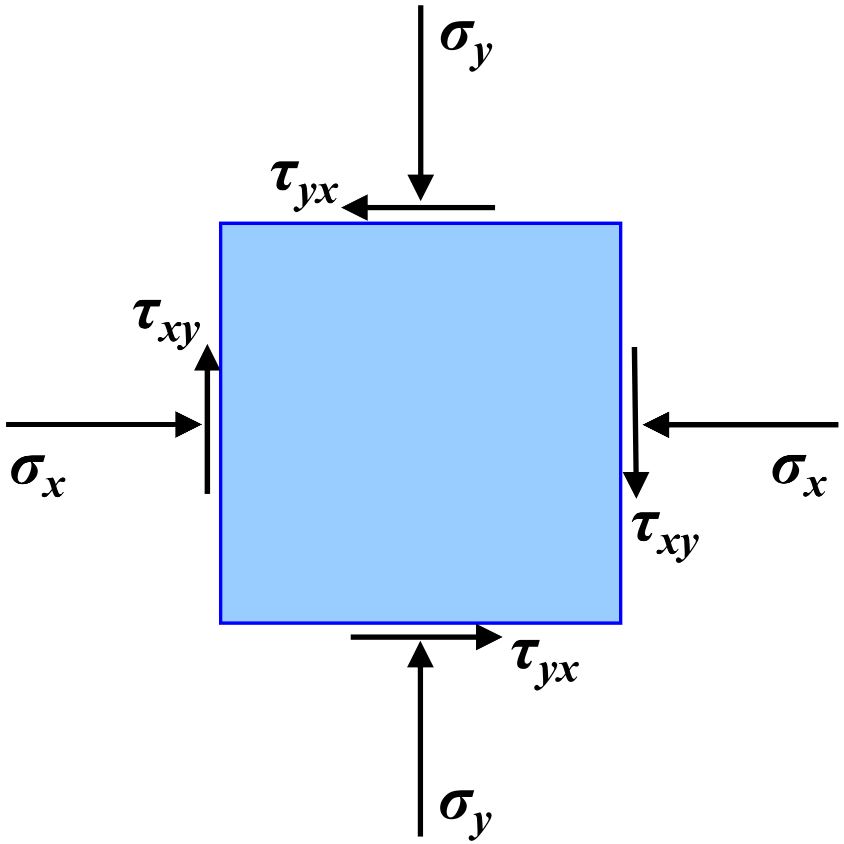


Figure shows *xy*-view of stresses acting on an elementary volume *dV*

As volume is at rest, no translational or rotational net force can be present:

$\tau_{xy} = \tau_{yx}$

Similar argument for *xz* and *yz*.

I.e., Stress tensor is symmetric

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

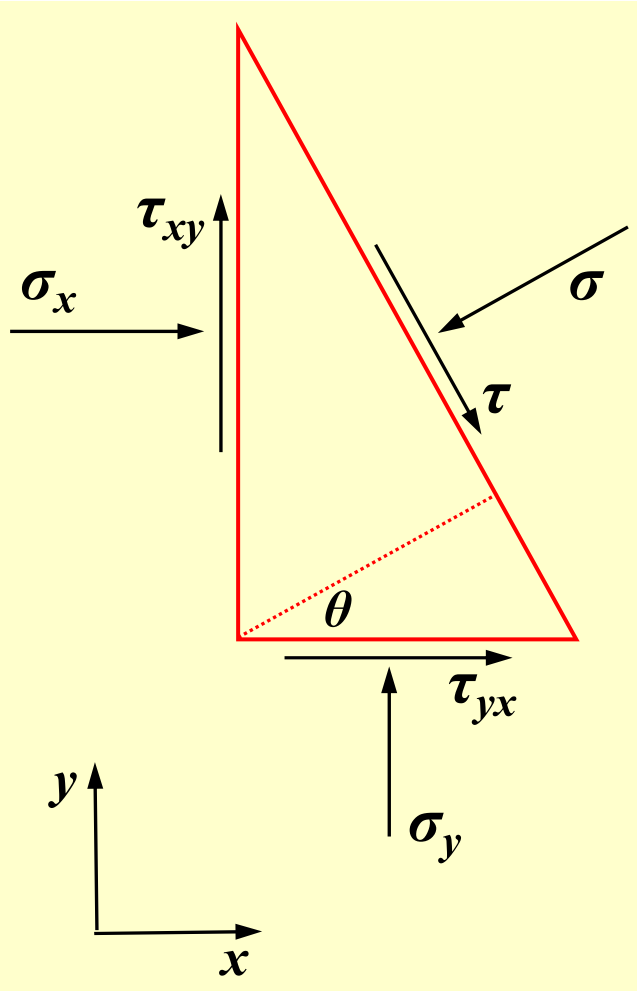
Hence the coordinate system can be rotated such that the stress tensor is diagonal in the rotated system:

$$\sigma^* = \mathbf{A}^T \sigma \mathbf{A} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Explicit Calculation 2D

Stress σ, τ acting on surface,
net forces on triangle at rest cancel:

Recall: Force F = σA, area of vertical = Acosθ; area of horizontal = Asinθ



$$\sigma A = A_y \sigma_x \cos\theta + A_x \sigma_y \sin\theta + A_y \tau_{xy} \cos\theta + A_x \tau_{xy} \sin\theta =$$
$$= A(\sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta)$$

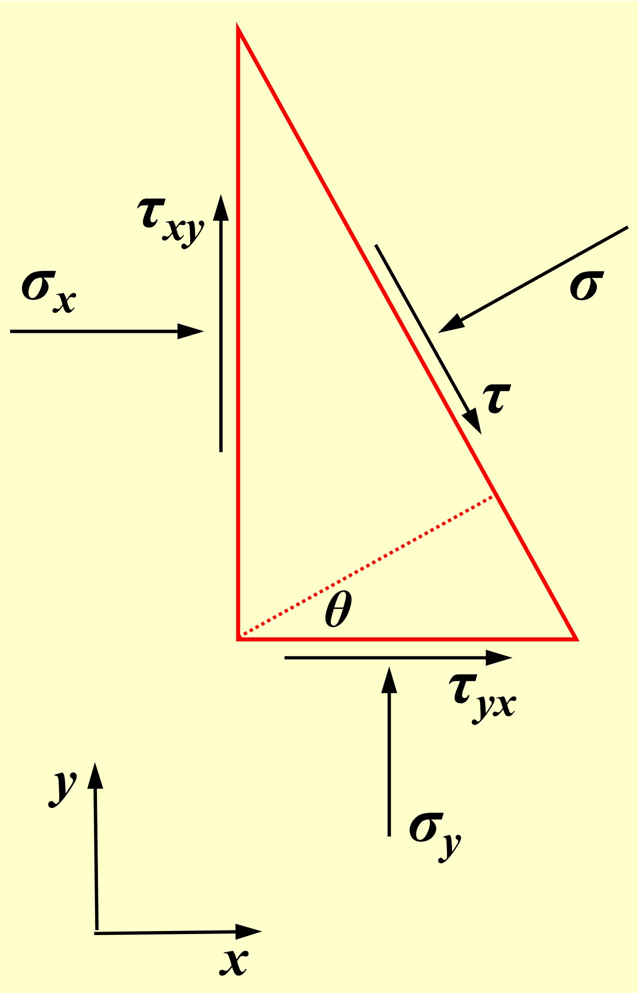
Hence, and similar for $\boldsymbol{\tau}$-direction:

$$\sigma = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$
$$\tau = \frac{1}{2}\left(\sigma_y - \sigma_x\right)\sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\tau = 0 \text{ when } \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Explicit Calculation 2D

$$\tau = 0 \text{ when } \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \Rightarrow \text{two solutions } \theta_1 \text{ and } \theta_2$$



The directions θ_1 and θ_2 for which τ vanishes are the principal axes of stress. (Orthogonal) Corresponding normal stresses σ_1 and σ_2 are principal stresses:

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \sqrt{\tau_{xy}^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2}$$

Explicit Calculation 2D

Using principal axes in previous formula

$$\sigma = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau = \frac{1}{2}\left(\sigma_y - \sigma_x\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

the shear stress terms vanish, and by straightforward manipulation:

$$\sigma = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta$$

$$= \frac{1}{2}\left\{\sigma_1(1-\sin^2\theta) + \sigma_2(1-\cos^2\theta) + \sigma_1\cos^2\theta + \sigma_2\sin^2\theta\right\}$$

$$= \frac{1}{2}\left\{\sigma_1 + \sigma_2 + (\sigma_1 - \sigma_2)\cos^2\theta + (\sigma_2 - \sigma_1)\sin^2\theta\right\}$$

$$= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta$$

$$\tau = \frac{1}{2}\left(\sigma_2 - \sigma_1\right)\sin 2\theta$$

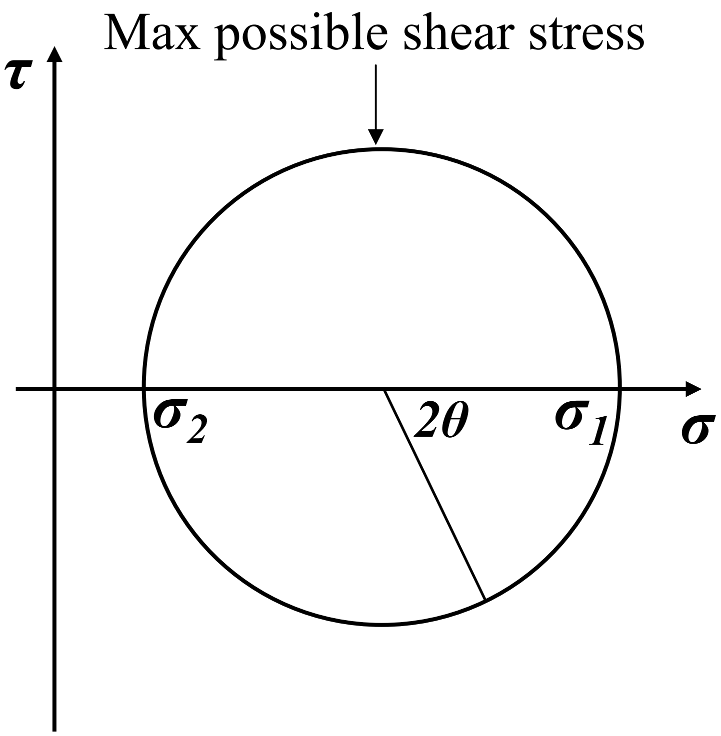
Explicit Calculation 2D

This can be expressed as:

$$\sigma - c = \quad r\cos 2\theta$$
$$\tau \qquad = -r\sin 2\theta$$

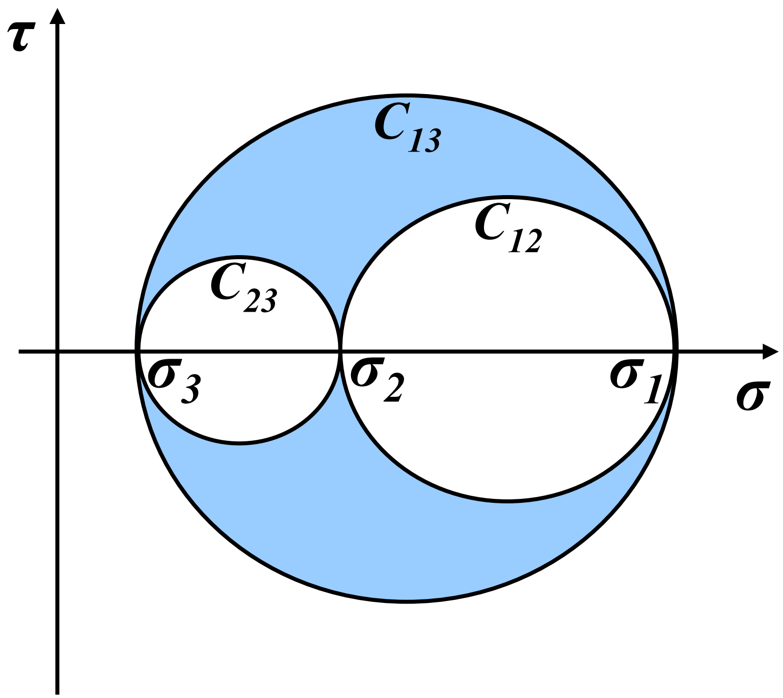
with

$$c = \frac{\sigma_1 + \sigma_2}{2} \text{ and } r = \frac{\sigma_1 - \sigma_2}{2}$$



I.e. a circle with centre at *(c,0)* and radius r.
It is called Mohr's (stress) circle.
Stresses σ and τ in any direction θ correspond to a point on the circle

What about 3D?



The construction of Mohr circles can be done in a similar manner in 3D. (A little more complicated obviously…)

When (σ, τ) in:

(x,y)-plane: (σ, τ) on circle C_{12}
(x,z)-plane: (σ, τ) on circle C_{13}
(y,z)-plane: (σ, τ) on circle C_{23}

else (σ, τ) in blue area between circles

Principal stress

When stress tensor is diagonal:

Axis system: Principal stress directions
Diagonal elements: Principal stresses; σ_1, σ_2, σ_3

As earth's surface is in contact with air or water
(which cannot support shear tractions),
the (horizontal) earth surface is a principal stress plane.

Hence one principal direction is vertical, while the other two act in an approximate horizontal plane.

→ In general also true in the upper crust, about 15-20 km down.

Principal stress (2)

Notation:
Vertical stress: σ_V
Maximum horizontal stress: $\sigma_{H,max}$, σ_H, σ_{H1}
Minimum horizontal stress: $\sigma_{H,min}$, σ_h, σ_{H2}

(*S* often used in place of σ)

Normally,

$$\sigma_V > \sigma_{H,max} > \sigma_{H,min}$$

A material will always fail in the direction of the largest stress, however small the difference may be. (E.g. breakout tests)

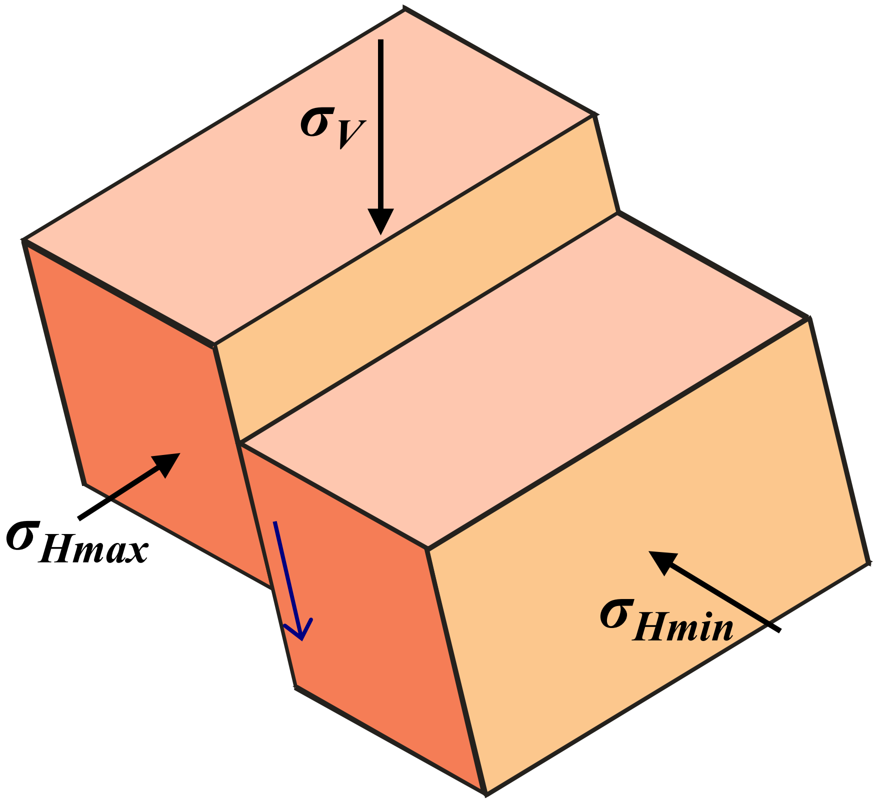
Rock failure & principal stress directions

We can clearly see the two perpendicular directions in which the rock has broken

Rock failure & principal stress directions

Another example showing the same feature.

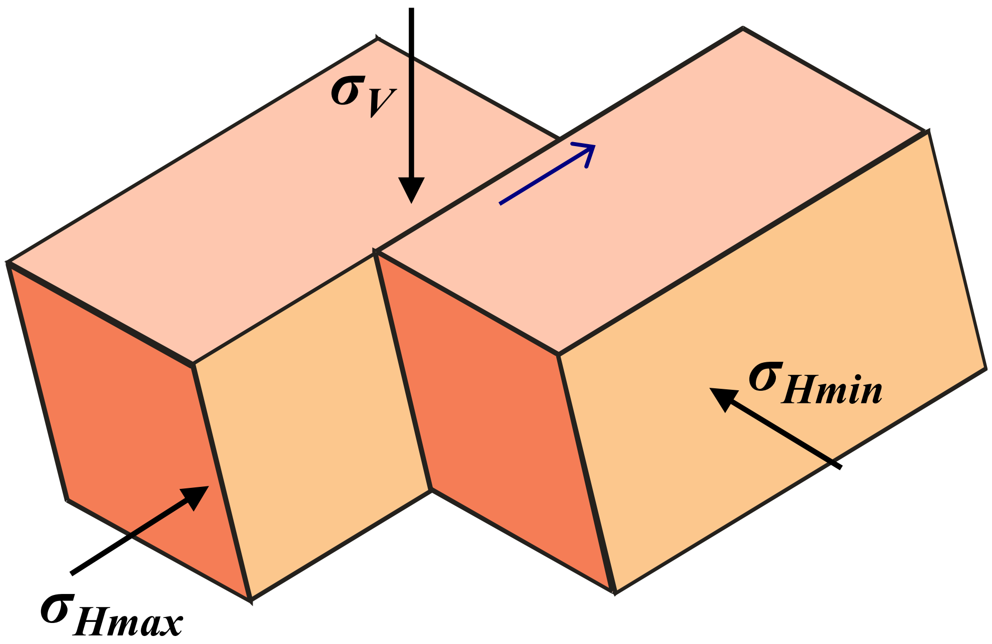
Stress ordering & Fault Types (Anderson)



Normal

$$\sigma_V > \sigma_{H,max} > \sigma_{H,min}$$

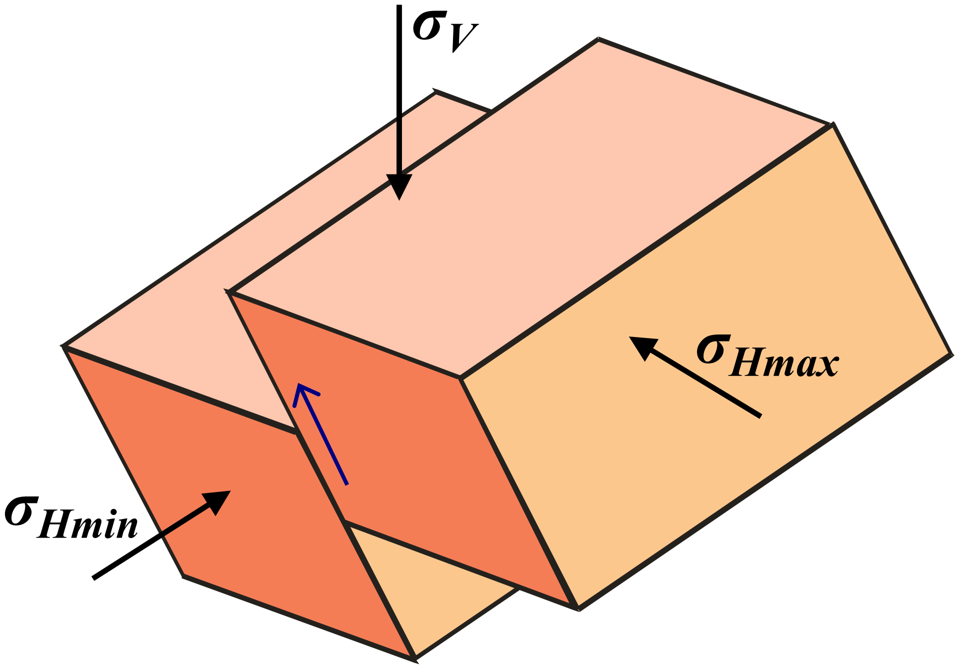
Stress ordering & Fault Types (Anderson)



Strike-slip

$$\sigma_{H,max} > \sigma_V > \sigma_{H,min}$$

Stress ordering & Fault Types (Anderson)



Reverse

$$\sigma_{H,max} > \sigma_{H,min} > \sigma_V$$

Other concepts

- Compressive stress is positive (by convention)
- Tensile stress (extension) is negative
- (Standard in rock mechanics, but not in solids mechanics)

- Examples of stress invariants (independent of coordinate system)
  - Mean normal stress:

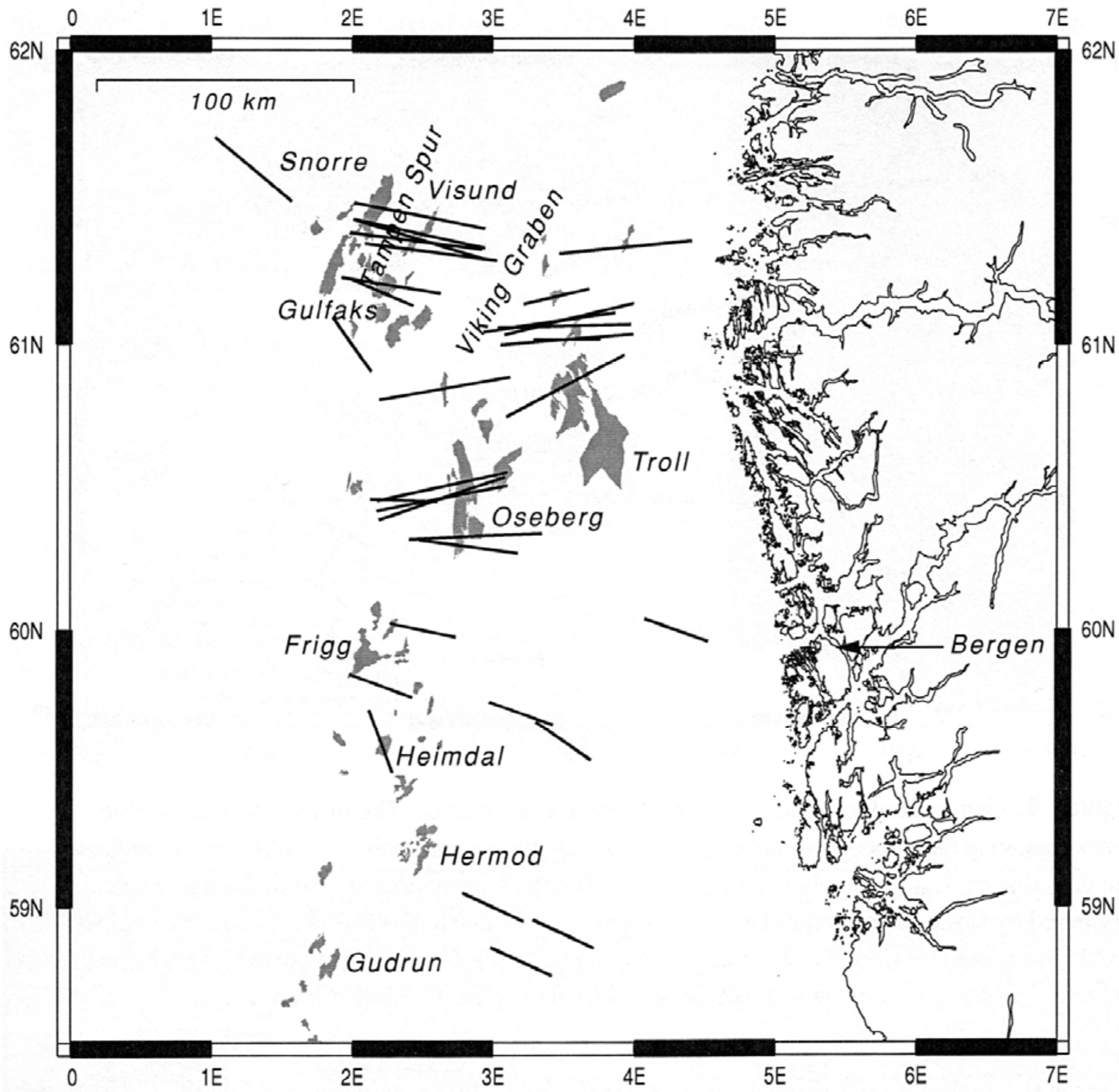
$$p = \bar{\sigma} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \quad (= \frac{1}{3} tr(\sigma))$$

  - Differential stress:

$$q = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}$$

$$= \sigma_1 - \sigma_3 = 2\tau \quad \text{when } \sigma_2 = \sigma_3 \quad \text{(axis symmentry)}$$

Principal stress directions – Global

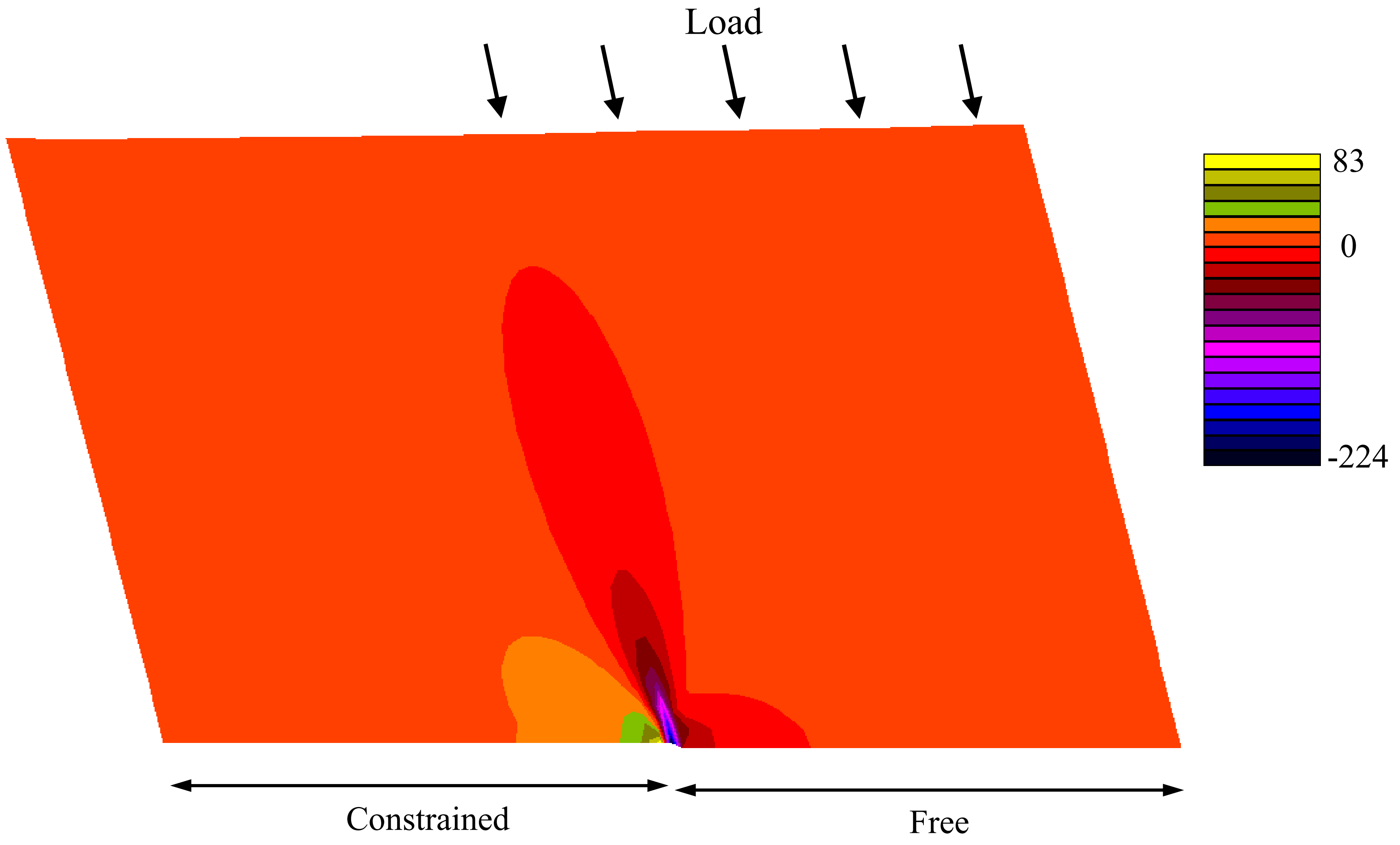


Direction & magnitude of σ_{Hmax} in northern North Sea.

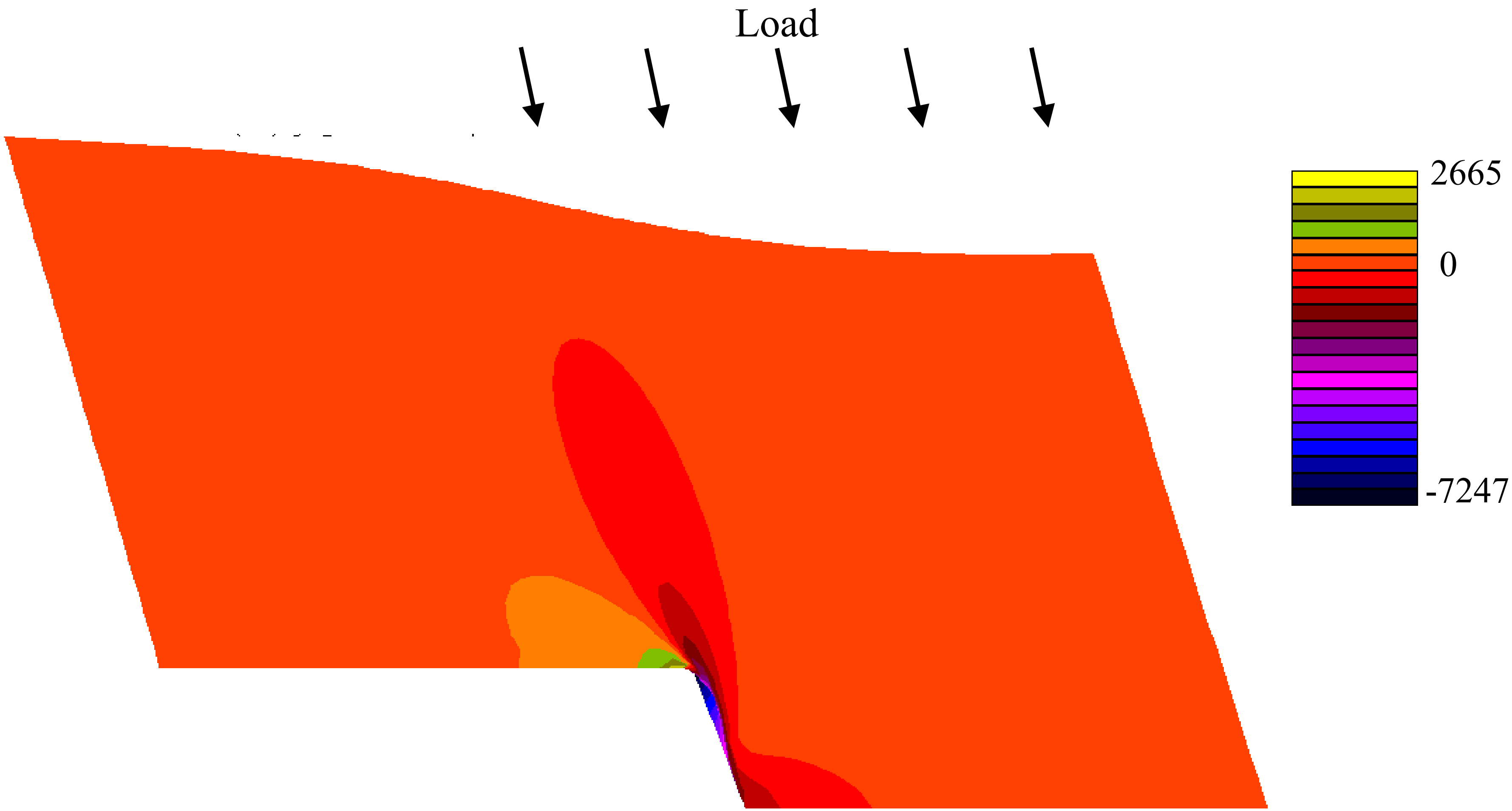
(from Zoback)

Determined by drilling induced tensile fractures and breakout tests in wells

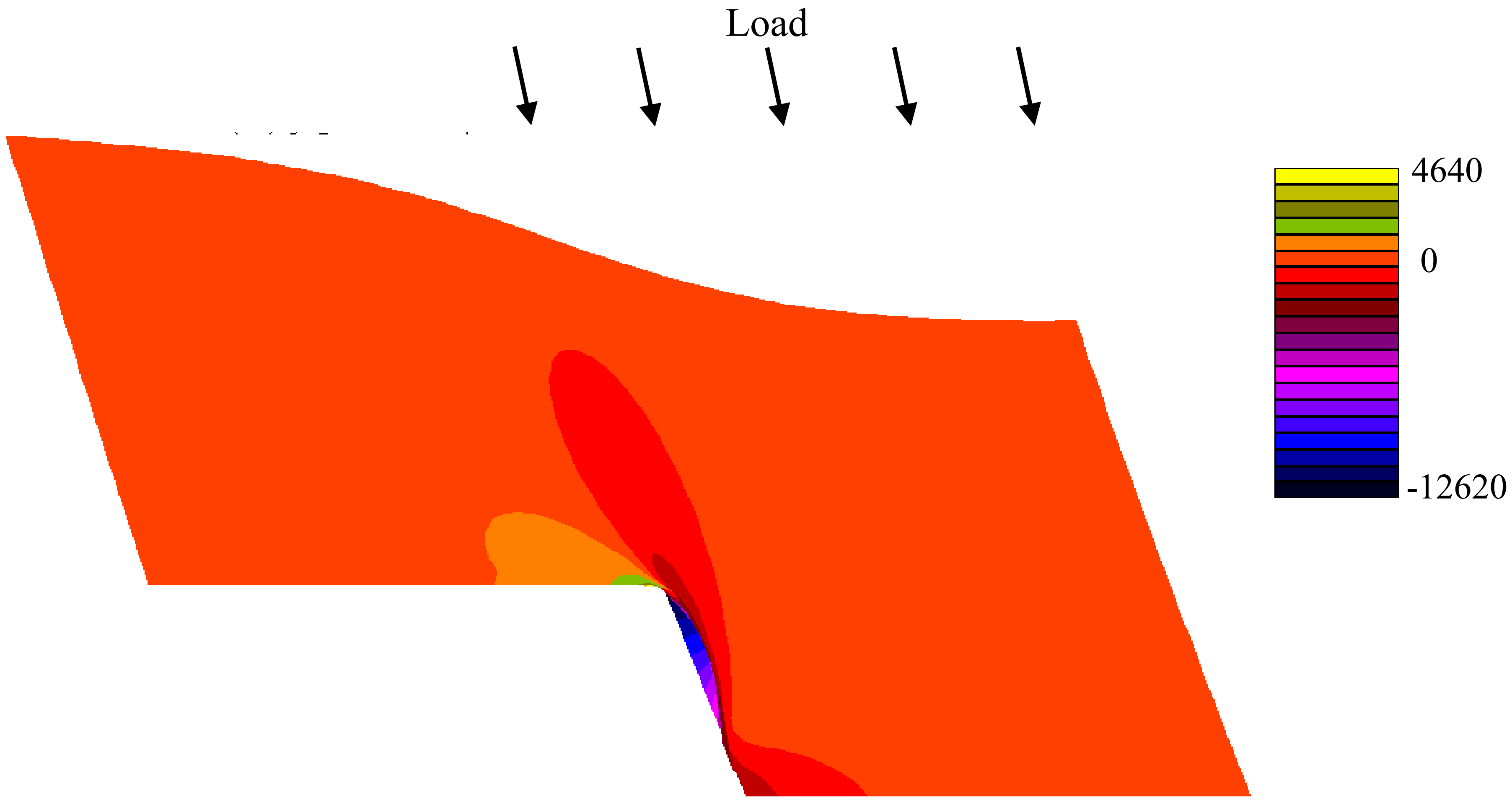
Local stress during fault generation (1)



Local stress during fault generation (2)

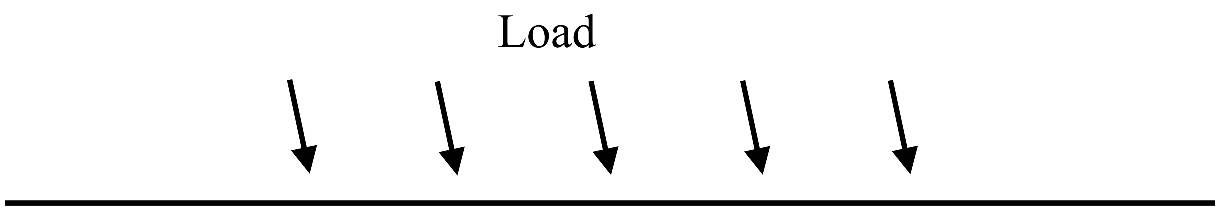


Local stress during fault generation (3)



→ Even though the global stress field is relatively uniform, (very) large local variations exist near irregularities as faults / fractures

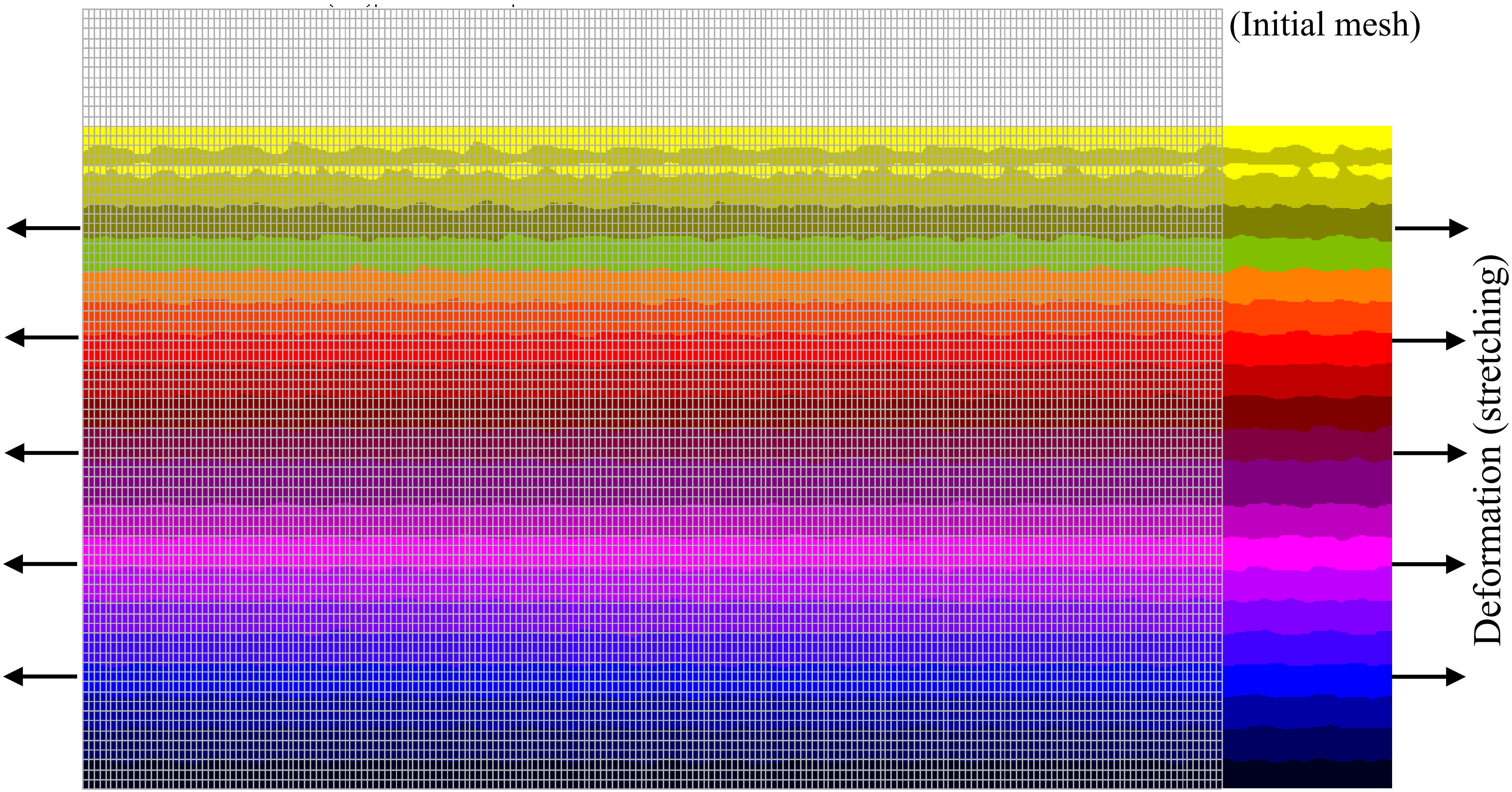
Stress irregularity



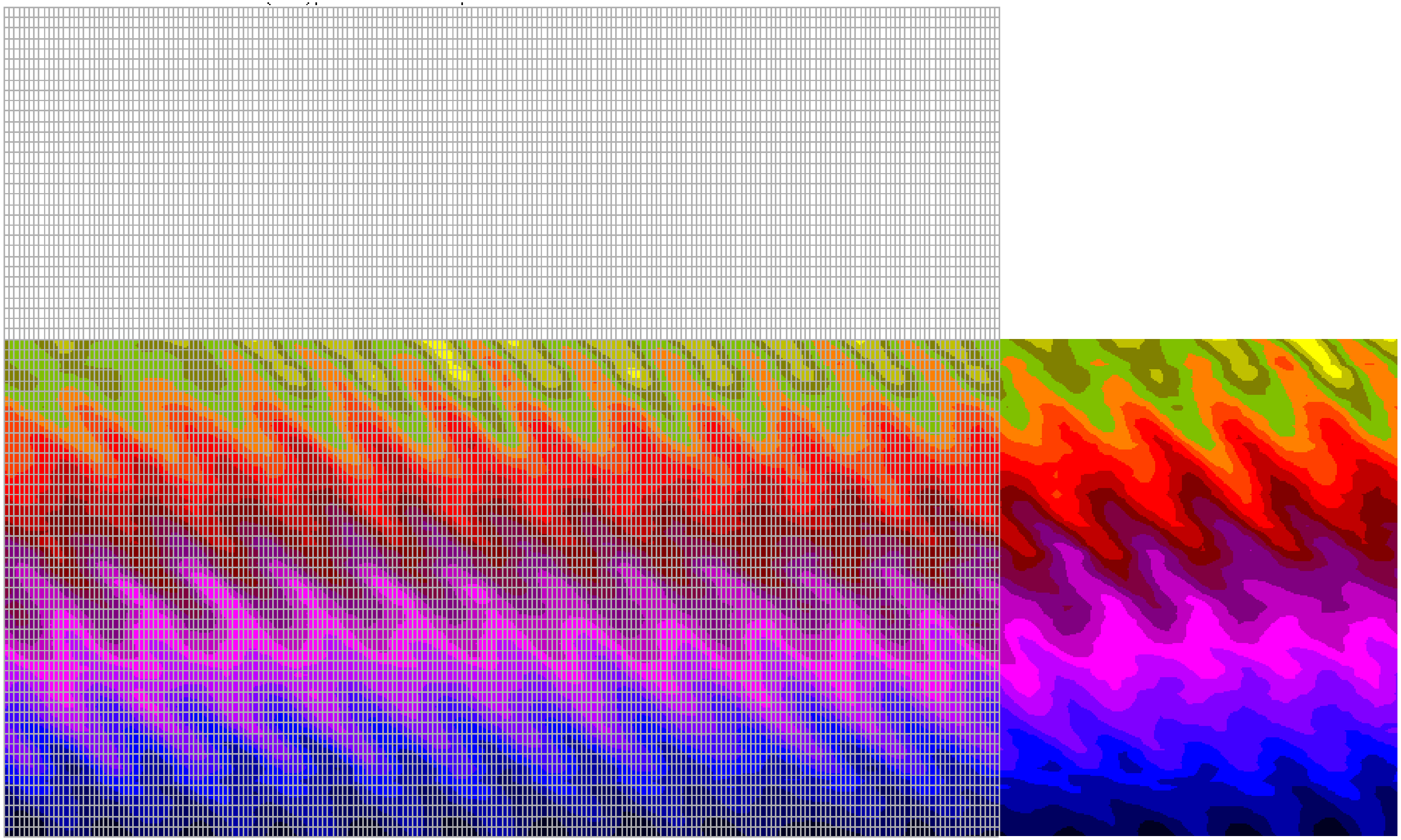
Uniform load on a homogeneous volume results in stiff motion or uniform deformation.
In order to kick off an irregular event, the load must have a local irregularity or the volume must have a local point of weakness.

Experiment: Simplistic model of Greenland drifting off from Scandinavia – resulting stress field in North Sea Basin.

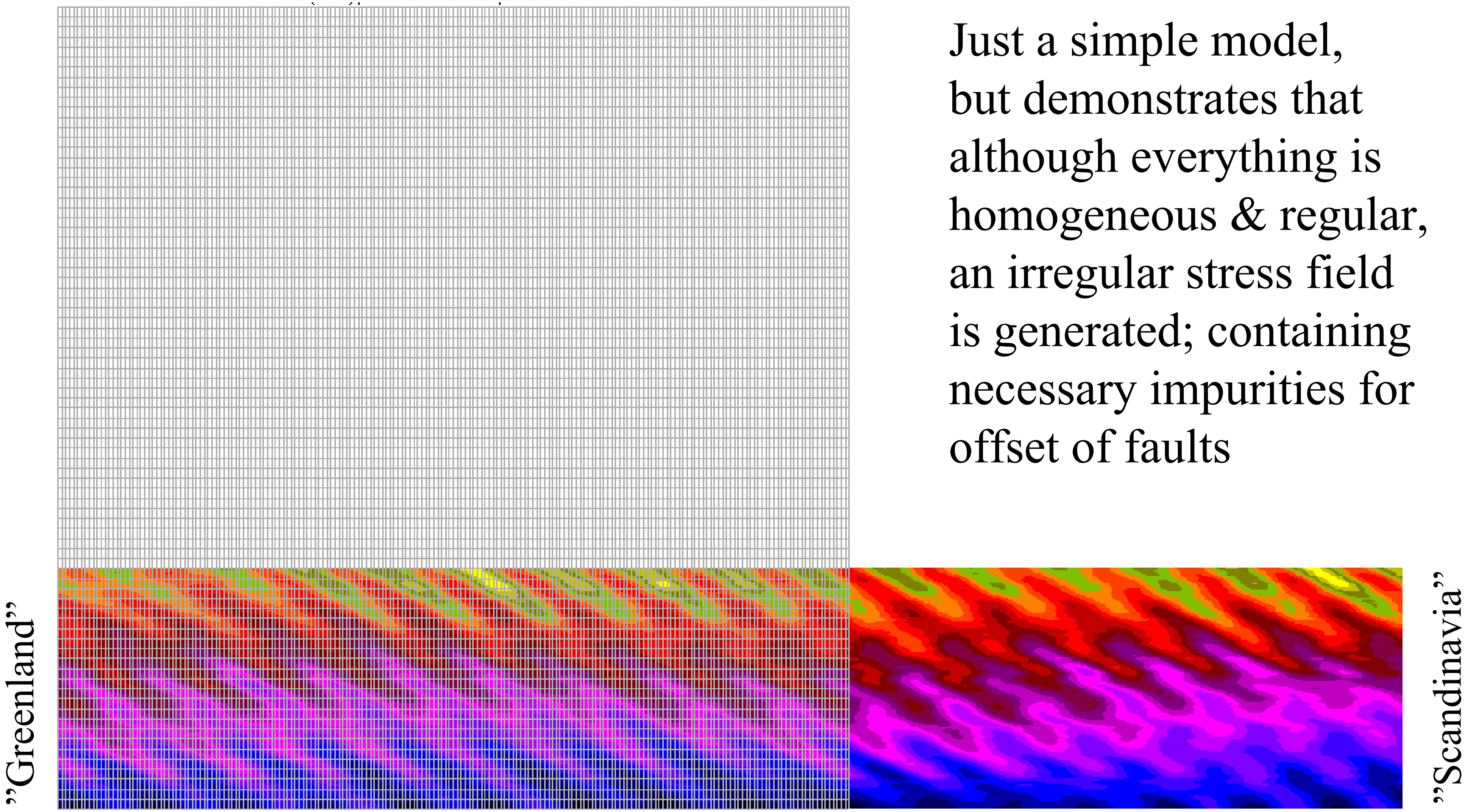
Stress in a volume under tension



Stress in a volume under tension (2)

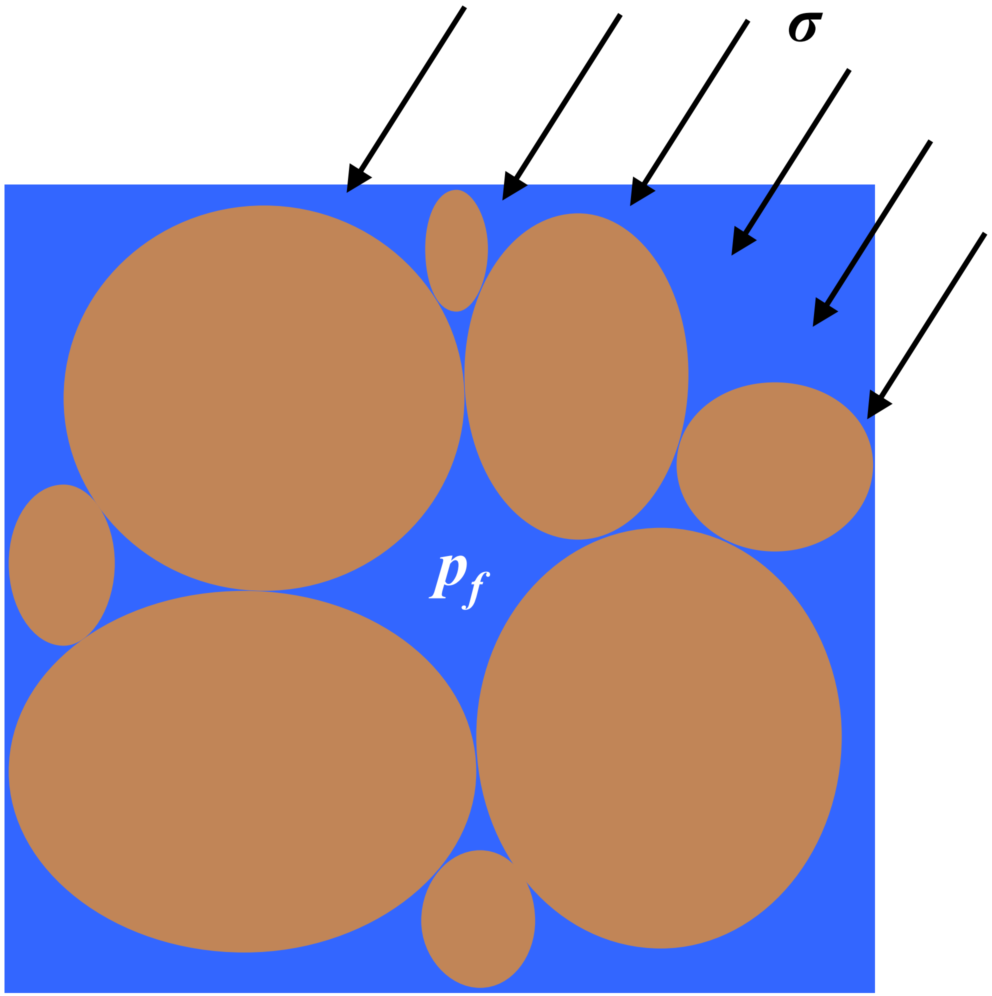


Stress in a volume under tension (3)



Just a simple model, but demonstrates that although everything is homogeneous & regular, an irregular stress field is generated; containing necessary impurities for offset of faults

Effective Stress – intuitive definition



In a porous rock, the main mechanism is deformation of the pore space, not the solid itself. The force acting on the pore walls is the external stress σ and an opposing force by the fluid pressure p_f.

The net force attempting to deform the pore wall is hence $\sigma' = \sigma - p_f$.

σ' is effective stress.

Effective Stress – actual definition

- In reality the grains will be somewhat compacted
- Pressure is a scalar, stress a tensor
- Taking account of both of these:

Effective stress σ':
(**I** is identity tensor)

$$\sigma' = \sigma - \alpha p_f \mathbf{I}$$

α is Biot's constant,

$$\alpha = 1 - \frac{\text{Grain compressibility}}{\text{Bulk compressibility}}$$

α always satisfies: $\varphi \le \alpha \le 1$
For sands / weak sandstone, $\alpha > 0.999$,
and hence normally set to 1.

Porous rock vs. solids

- In a solid, the relevant external force is the applied stress
- Strength properties of a solid is tied to the solid itself
- Strength and deformation of a porous rock
  - is only to a small degree dependent on the solid (grains)
  - is dependent on the strength of the pore walls
  - deformation means deformation of void space, not grains
- Hence: For porous rock / soil, effective stress is the governing parameter
- (Much of the theory to follow was developed for solids, and may need rethinking before applied to porous rocks)